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# The Aharonov-Bohm effect around a rotating black hole analogue

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# Outline

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- In the QM context, a relative phase difference $\Delta\phi$ is acquired by paths with the same start and end points that pass in opposite sides of the solenoid, to be $\Delta\phi = \frac{q\Phi}{\hbar c} = 2\pi\alpha.$
- This effect will exist, even though there are no magnetic fields acting in the regions where the electron beam passes, suggesting that $\mathbf{A}$ has a certain physical significance. [Chambers, 1960] confirmed experimentally the Aharonov-Bohm (AB) effect.

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- In [Berry *et. al*, 1980] it is shown that if we have gravity waves on water in a locally irrotational fluid flow with a vortex in the center, it will experience an analogue AB effect.
- In this case, instead of $\mathbf{A}$, we will have: $\nabla \times \boldsymbol{v}_0 = \mathbf{0}$, where $\boldsymbol{v}_0$ is the background flow velocity.

# Draining Bathtub Vortex

- [Unruh, 1981] has presented a new way to see gravitation effects by studying analogue models in fluid dynamics.
- Starting with some conditions, as inviscid fluid, irrotational, and barotropic flows, the movement of these fluids is divided in two parts: background flow ($\boldsymbol{v}_0$) + perturbations in the flow ($\delta\mathbf{v} = -\nabla\psi$).
- The potential perturbation $\psi$ obeys the Klein-Fock-Gordon (KFG) equation, with an effective metric $g_{\mu\nu}$ that "mimics" curved spacetimes.

$$\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\psi\right) = 0. \tag{1}$$

# Draining Bathtub Vortex

- The model for a rotating black hole analogue was first developed by [Visser, 1998].
- In this model, $g_{\mu\nu}$ are given implicitly by the infinitesimal interval $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$:

$$ds^2 = c_s^2 d\tilde{t}^2 - \left(dr + \frac{Dd\tilde{t}}{r}\right)^2 - \left(rd\tilde{\phi} - \frac{Cd\tilde{t}}{r}\right)^2, \qquad (2)$$

  where $c_s^2 = a_g h_\infty$ ($a_g$ being the gravitational acceleration and $h_\infty$ the height of fluid far from the scattering center). The constants $C$ and $D$ are circulation and draining rates, respectively.
- The background has the profile:

$$\boldsymbol{v}_0 = -\frac{D}{r}\hat{r} + \frac{C}{r}\hat{\tilde{\phi}}. \qquad (3)$$

# Draining Bathtub Vortex

- Setting $c_s = 1$, we provide the change of the coordinates

$$d\tilde{t} \to dt = d\tilde{t} - \frac{D}{rf} dr$$

and

$$d\tilde{\phi} \to d\phi = d\tilde{\phi} - \frac{CD}{r^3 f} dr,$$

with $f = 1 - D^2/r^2$.

- Then, Eq. (2) is rewrite as:

$$ds^2 = \left(1 - \frac{D^2 + C^2}{r^2}\right) dt^2 - f^{-1} dr^2 - 2C d\phi dt - r^2 d\phi^2. \quad (4)$$

- In this system of coordinates, it is easy found that the event horizon is located at $r_h = D$ and the ergo-region is within $r_e = \sqrt{C^2 + D^2}$.

# Solutions to the KFG Equation

- To solve Eq. (1), we use the *ansatz*

$$\psi(t,r,\phi) = R_{m\omega}(r)e^{i(m\phi-\omega t)}/\sqrt{r}, \tag{5}$$

being $\omega$ the angular frequency of monochromatic planar waves.

- The equation to $R_{m\omega}$ is

$$\frac{d^2}{dr_*^2}R_{m\omega}+\left[\left(\omega-\frac{Cm}{r^2}\right)^2-\frac{f}{r^2}\left(m^2-\frac{1}{4}+\frac{5D^2}{4r^2}\right)\right]R_{m\omega}=0, \tag{6}$$

with $dr_* = dr/f$, so that when $r\to\infty$, $r_*\to r$.

# Solutions to the KFG Equation

- The exact solution to Eq. (6) for $m \neq 0$ is given in terms of Heun functions [Vieira and Bezerra, 2014], whose properties are not well known yet.
- However, it is possible to find analytic solutions to Eq. (6) for a slowly rotating acoustic hole ($C/r$ small).
- In this regime, the solutions we seek far from the horizon and close to it are given by:

$$R_{m\omega} \approx \begin{cases} \sqrt{\pi\omega r_*/2}\left[A_m^{(\text{in})} e^{-i(\nu+1/2)\pi/2} H_\nu^{(2)}(\omega r_*) + \right. \\ \left. \quad + A_m^{(\text{out})} e^{i(\nu+1/2)\pi/2} H_\nu^{(1)}(\omega r_*)\right], r \gg r_h, \\ \exp\left[-i\left(\omega - mC/D^2\right) r_*\right], \qquad r \gtrapprox r_h, \end{cases} \tag{7}$$

with $A_m^{(\text{in/out})}$ being constants and $H_\nu^{(1/2)}(x)$ the Hankel functions.

# Partial-Wave Method

- The partial-wave method in two spatial dimensions is summarized in [Lapidus, 1982]. This method consists in writing the solution of Eq. (1) in a series form:

$$\psi(t, r, \phi) = e^{-i\omega t} \sum_{m=-\infty}^{+\infty} R_{m\omega}(r) e^{im\phi}/\sqrt{r}. \qquad (8)$$

- Far away from the scatter center, the wave function can be expressed as

$$\psi(t, r, \phi) = e^{-i\omega t} \left( e^{i\omega x} + f_\omega(\phi) e^{i\omega r_*}/\sqrt{r} \right), \qquad (9)$$

where $f_\omega(\phi)$ is the scattering amplitude, and it was considered a plane wave propagating in the $+x$ direction.

# Partial-Wave Method

- With the Eqs. (8), (9) and the asymptotic expression of $R_{m\omega}$ for $r \gg r_h$ in Eq. (7), we find:

$$f_\omega(\phi) = (2\pi i\omega)^{-1/2} \sum_{m=-\infty}^{+\infty} (e^{2i\delta_m} - 1)e^{im\phi}. \qquad (10)$$

- The phase shifts $\delta_m$ are given by [Dolan and Oliveira, 2013]:

$$e^{2i\delta_m} = i(-1)^m A_m^{(\text{out})}/A_m^{(\text{in})}, \qquad (11)$$

and $A_m^{(\text{in})} = (-1)^m(-2\pi i\omega)^{-1/2}$.

# Phase shifts

- In the case $m = 0$, Eq. (6) has a simple solution in terms of Bessel function:

$$R_{m\omega}(r) = A\sqrt{r}J_{-i\omega D}(\omega r\sqrt{f}). \tag{12}$$

- We find that $A = e^{-\omega D\pi/2}$ and $\delta_{m=0} = i\pi\omega D/2$.
- For large values of $|m| \gg \omega\sqrt{C^2 + D^2}$, it is used the Born approximation (in two spatial dimensions see [Adhikari and Hussein, 2008]). As shown in [Dolan, Oliveira and Crispino, 2011], these values of $\delta_m$ are:

$$\delta_m \approx -\frac{\alpha\pi}{2}\text{sgn}(m) + \frac{3\pi(\alpha^2+\beta^2)}{8|m|} - \frac{5\alpha\pi(\alpha^2+\beta^2)}{8m^2}\text{sgn}(m), \tag{13}$$

being $\text{sgn}(m)$ the signal function of $m$ and $\alpha = \omega C$, $\beta = \omega D$.

# Semi-classical interpretation: Geodesics Analysis

- Via geodesics analysis [Dolan and Oliveira, 2013], we find the deflection angle $\Theta$ of null geodesics:

$$\Theta(\tilde{b}) = \frac{3\pi(C^2 + D^2)}{4\tilde{b}^2} - \frac{5\pi C(C^2 + D^2)}{2\tilde{b}^3} + \mathcal{O}(\tilde{b}^{-4}), \qquad (14)$$

  where $\tilde{b} = L/E$ ($L$ and $E$ are constants corresponding to angular momentum and energy of phonons, respectively).
- Eq. (14) is related to Eq. (13) by the semi-classical relation

$$\Theta(\tilde{b}) = -d(2\delta_m)/dm$$

  with $m/\omega \to \tilde{b}$.

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  with $m/\omega \to \tilde{b}$.
- Then, the last two terms of Eq. (13) are associated to phonon deflection (symmetric and asymmetric). The first one is due to a relative time difference accrued by geodesics passing on opposite sides of the vortex [Dolan, Oliveira and Crispino, 2011].

# Analogue AB Effect - $\alpha\beta$ Effect

- In [Lapidus, 1982, Adhikari and Hussein, 2008] we find the expression for the "scattering cross section":

$$\frac{d\sigma}{d\phi} = |f_\omega(\phi)|^2 . \tag{15}$$

- To low frequency scattering, it is enough to take only the first term of (13): $\delta_m = -\frac{\alpha\pi}{2}\mathrm{sgn}(m)$ and $\delta_{m=0} = i\pi\beta/2$.
- With them we get the scattering cross section of the $\alpha\beta$ effect:

$$\frac{d\sigma_{\alpha\beta}}{d\phi} = \frac{\pi}{2\omega}\left[\alpha\frac{\cos(\phi/2)}{\sin(\phi/2)} - \beta\right]^2 + \mathcal{O}(\omega^2). \tag{16}$$

# Analogue AB Effect - $\alpha\beta$ Effect

- We compare the scattering cross section of the $\alpha\beta$ effect, given by (16), with to the AB effect [Aharonov and Bohm, 1959]:

$$\frac{d\sigma_{AB}}{d\phi} = \frac{1}{2\pi\omega}\frac{\sin^2(\pi\tilde{\alpha})}{\sin^2(\phi/2)}, \quad (17)$$

where $\tilde{\alpha} = |e|\Phi/ch$, $\Phi$ is the magnetic flux.

- With $\beta = 0$ (only vortex) both effects are very similar; they are symmetric by $\phi \to -\phi$, but in the $\alpha\beta$ effect, $\frac{d\sigma_{\alpha\beta}}{d\phi} = 0$ to $\phi \to 180°$.
- As we increase the draining rate, we see that both effects begin to become more different to each other.

# Analogue AB Effect - $\alpha\beta$ Effect

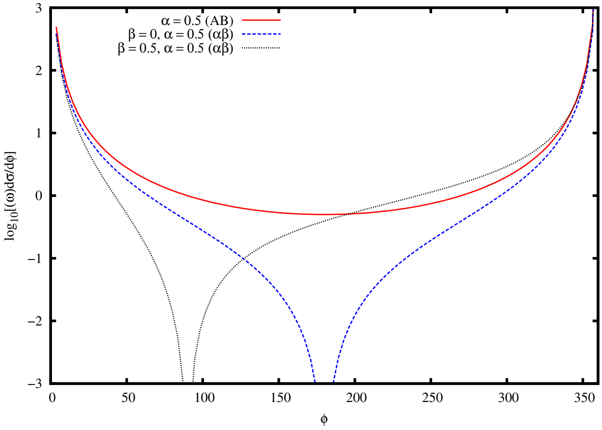


Figure: Analysis of differential scattering cross section of the AB effect and $\alpha\beta$ effect, with $\tilde{\alpha} = \alpha = 0.5$.

# Analogue AB Effect - $\alpha\beta$ Effect

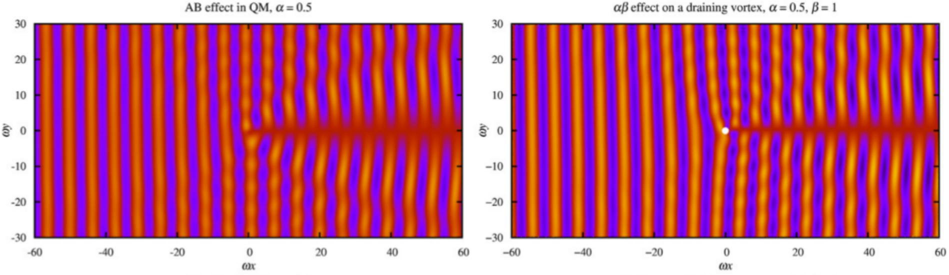


Figure: The left plot shows the AB effect near an infinitesimal solenoid, and the right plot shows the "$\alpha\beta$ effect" near a draining vortex with $\beta = 1$ and $\alpha = 0.5$, illustrating wave front intersection along a seam. (Extracted from [Dolan and Oliveira, 2013].)

# Remarks

- We have discussed about the AB effect in a draining bathtub vortex.
- It was found analytically $\delta_{m=0}$ and $\delta_m$ using the Born approximation for larges values of $m$.
- In the low-frequency regime, we found the scattering cross section to the "$\alpha\beta$ effect" and compared it with the original AB effect.

# References


- Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).
- R. G. Chambers, Phys. Rev. Lett. **5**, 3 (1960).
- M. V. Berry, R. G. Chambers, M. D. Large, C. Upstill, J.C. Walmsley, Eur. J. Phys. **1**, 154 (1980).
- W. G. Unruh, Phys. Rev. Lett. **46**, 1351 (1981).
- M. Visser, Class. Quantum Grav. **15**, 1767 (1998).
- C. Barceló, S. Liberatti and M. Visser, `arXiv:gr-qc/0505065v3` (2011).
- H. S. Vieira and V. B. Bezerra, `arXiv:1406.6884` (2014).
- I. R. Lapidus, Am. J. Phys. **50**, 45 (1982).

# References


- S. R. Dolan, E. S. Oliveira and L. C. B. Crispino, Phys. Lett. B **701**, 485 (2011).
- S. R. Dolan and E. S. Oliveira, Phys. Rev. D **87**, 124038 (2013).
- S. K. Adhikari and M. S. Hussein, Am. J. Phys. **76**, 1108 (2008).
- E. Berti, V. Cardoso and J. P. S. Lemos, Phys. Rev. D **70**, 124006 (2004).

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Thanks for your attention.