# On Black Hole Structures in Scalar-Tensor Theories of Gravity

Júlio C. Fabris

Departamento de Física - Universidade Federal do Espírito Santo - Brasil

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# Black holes in General Relativity

## The types

- There are essentially four kind of black hole solutions in the General Relativity theory.
    1. Schwarzschild - a unique parameter, the mass.
    2. Reissner-Nordström - two parameters, mass and charge.
    3. Kerr - two parameters, mass and angular momentum.
    4. Kerr-Newmann - three parameters, mass, charge and angular momentum.

# Black holes in General Relativity

## Static solutions

- The Schwarzschild solution:

$$ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 - r^2 d\Omega^2.$$

- The Reissner-Nordström solution:

$$ds^2 = \left(1 - \frac{2GM}{r} + \frac{Q^2}{r^2}\right)dt^2 - \left(1 - \frac{2GM}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 - r^2 d\Omega^2.$$

# Black holes in General Relativity

## Static solutions: Schwarzschild

- The Schwarzschild solution is characterised by a unique parameter: The mass $M$.
- The horizon occurs at

$$R_H = 2GM.$$

- The Hawking temperature is given by,

$$T_H = \frac{1}{8\pi}\frac{1}{M}.$$

- The temperature decreases as the mass (energy) increases.

# Black holes in General Relativity

## Static solutions: Schwarzschild

- The entropy is given by,

$$S_{BH} = 4\pi M^2 = \frac{A_{BH}}{4}.$$

- The heat capacity is negative:

$$C_{BH} = -8\pi M^2.$$

# Black holes in General Relativity

## Static solutions: Reissner-Nordström

- The Reissner-Nordström solution is characterised by two parameters: The mass $M$ and the charge $Q$.
- The horizon occurs at,

$$R_{\pm} = M \pm \sqrt{M^2 - Q^2}.$$

- There are two horizons:
  - The event horizon, at $R_{+} = M + \sqrt{M^2 - Q^2}$.;
  - The Cauchy horizon, at $R_{-} = M - \sqrt{M^2 - Q^2}$.

# Black holes in General Relativity

## Static Solutions: Reissner-Nordström

- The horizons become degenerate for $M = Q$, the extremal black hole.
- The horizons disappear for $Q > M$, forming a naked singularity.

# Black holes in General Relativity

## Static solutions: Reissner-Nordström

- The temperature of the Reissner-Nordström black hole is given by,

$$\begin{aligned} T_{RN} &= \frac{1}{8\pi M}\left(1 - 16\pi^2\frac{Q^4}{A^2}\right), \\ A &= 4piR_+^2. \end{aligned}$$

- The temperature of the Reissner-Nordström black hole becomes zero when the extremal condition is satisfied, $M = Q$, and it becomes negative for the naked singularity case.

# Black holes in General Relativity

## Static solutions: Reissner-Nodström

- The entropy of the Reissner-Nordström black holes is still given by the the area law:

$$S = \pi R_+^2 .$$

- The entropy is always finite, including for the extremal case (remember talk by José Lemos).
- But, it becomes ill defined for the naked singularity case.

# Black holes in General Relativity

## General cases

- Considering rotation, a new parameter is introduced: the angular velocity.
- In the rotating case there is also the extremal condition.
- There are naked singularity cases.
- In fact, in General Relativity, all asymptotically flat black holes are characterised in principle at most by three parameters: mass, charge and angular momentum.
- This fact constitute the basis of the *no-hair theorem*.
- But, remember the C. Herdeiro talk!

# Scalar fields

## The different couplings

- The introduction of scalar fields are the simplest extension of General Relativity.
- It adds a spin 0 degree of freedom to the theory.
- We can think on some different possibility to introduce a scalar field.

# Scalar field

## The minimal coupling

- The minimal coupling is defined by the Lagrangian,

$$\mathcal{L} = \sqrt{-g}\biggl\{R - \epsilon\phi_{;\rho}\phi^{;\rho} + 2V(\phi)\biggr\}.$$

- In this Lagrangian, $V(\phi)$ is the potential self-interacting term.
- This kind of theory in general satisfy the energy conditions if $\epsilon = 1$ (ordinary field) and violate the energy conditions if $\epsilon = -1$ (phantom fields).

# Scalar field

## Non-minimal coupling

- We may have a much less trivial configuration using the non-minimal coupling.

$$\mathcal{L} = \sqrt{-g}\left\{f(\phi)R - \omega(\phi)\frac{\phi_{;\rho}\phi^{;\rho}}{\phi} + 2V(\phi)\right\}.$$

where $f(\phi)$ and $\omega(\phi)$ are, in general, functions of the scalar field itself.

- The case,

$$f(\phi) = \phi, \quad \omega(\phi) = \text{constant}, \quad V(\phi) = 0,$$

defines the traditional Brans-Dicke theory, the prototype of the scalar-tensor theories.

# Scalar fields
## $K$-Essence models

- Another possibility of including scalar field is a non-canonical kinetic term, leading to the $K$-Essence models.

$$\mathcal{L} = \sqrt{-g}\biggl\{R - \omega(\phi)f(X) + 2V(\phi)\biggr\},$$

where,

$$X = \phi_{;\rho}\phi^{\rho},$$

and $f$ is a given function.

# Scalar fields

## Other possibilities

- Other possibilities are the Horndeski theory, which is the most general scalar-tensor Lagrangian leading to second order differential equations.
- Or yet, the Galileons theory, where a scalar field exhibits some special translational symmetry.
- One example of such generalisation is the theory defined by,

$$\mathcal{L} = \sqrt{-g}\biggl\{R - G^{\mu\nu}\phi_{;\mu}\phi_{;\nu} + 2V(\phi)\biggr\},$$

where $G_{\mu\nu}$ is the Einstein tensor.

# Black holes in minimally coupled models

## No black hole for ordinary scalar field

- For the ordinary case where the scalar field is minimally coupled to gravity, without potential, represented by the equations

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= \phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi_{;\rho}\phi^{;\rho}, \\ \Box\phi &= 0, \end{aligned}$$

there is no black hole solution.

# Black holes in minimally coupled models

## No black hole for ordinary scalar field

- In fact, let us write the four-dimensional spherically symmetric, static metric as,

$$ds^2 = e^{2\gamma}dt^2 - e^{2\alpha}du^2 - e^{2\beta}d\Omega^2,$$

where the $\alpha$, $\beta$ and $\gamma$ are functions only of the radial coordinate $u$.

# Black holes in minimally coupled models

## No black hole for ordinary scalar field

- The field equations are:

$$
\begin{aligned}
-2\beta'' - 3\beta'^2 + 2\beta'\alpha' + e^{2(\alpha-\beta)} &= \frac{\epsilon}{2}\phi'^2, \\
2\gamma'\beta' + \beta'^2 - e^{2(\alpha-\beta)} &= \frac{\epsilon}{2}\phi'^2, \\
\beta'' + \gamma'' + \beta'(\gamma' + \beta' - \alpha') + \gamma'^2 - \alpha'\gamma' &= -\frac{\epsilon}{2}\phi^2, \\
\phi'' + [\gamma' + 2\beta' - \alpha']\phi' &= 0.
\end{aligned}
$$

- For an ordinary scalar field, $\epsilon = 1$.

# Black holes in minimally coupled models

## No black hole for ordinary scalar field

- Choosing the coordinate condition,

$$\alpha = \gamma + 2\beta,$$

The equation for the scalar field can be easily solved:

$$\phi = \phi_0 u + \phi_1.$$

# Black holes in minimally coupled models

## No black hole for ordinary scalar field

- The remaining equations simplify to,

$$\begin{aligned}
-2\beta'' + \beta'^2 + 2\beta'\gamma' + e^{2(\gamma+\beta)} &= \frac{\epsilon}{2}\phi'^2, \\
2\gamma'\beta' + \beta'^2 - e^{2(\gamma+\beta)} &= \frac{\epsilon}{2}\phi'^2, \\
\beta'' + \gamma'' + \beta'^2 - 2\beta'\gamma' &= -\frac{\epsilon}{2}\phi^2.
\end{aligned}$$

# Black holes in the minimally coupled models

## No black hole for ordinary scalar field

- A combination of those equations lead to,

$$\gamma'' = 0.$$

- The solution is:

$$\gamma = bu.$$

# Black holes in the minimally coupled models

## No black hole for ordinary scalar field

- The final metric is:

$$ds^2 = P^a dt^2 - P^{-a} du^2 - P^{1-a} u^2 d\Omega^2,$$

with

$$P(u) = 1 - 2\frac{k}{u} \quad , \quad \phi = -\frac{C}{2k} \ln P(u),$$
$$\frac{m}{k} = a \quad , \quad a^2 = 1 - \frac{C^2}{2k^2}.$$

- Clearly, there is a naked singularity at $u = 2k$: it is a center.

# Black holes in the minimally coupled models

## Black hole with phantom fields

- The situation changes drastically if the null energy condition is violated.
- The null energy condition requires that

$$\rho + p \geq 0.$$

- In order to violate this condition, the kinetic term must appear with the *wrong* sign.

# Black holes in the minimally coupled models

## Black hole with phantom fields

- If $\epsilon = -1$, a phantom field, we have,

$$\begin{aligned} ds^2 &= P^a dt^2 - P^{-a} du^2 - P^{1-a} u^2 d\Omega^2 , \\ a^2 &= 1 + \frac{C^2}{2k^2} > 1. \end{aligned}$$

- This case represents a black hole, with an event horizon at $m = 2k$.
- Two remarkable features:
  1. Infinity event horizon hypersurface;
  2. Zero surface gravity.

# Non-minimal coupling

## The general action

- We can write a general action with a non-minimal coupling as,

$$\mathcal{L} = \sqrt{-g}\biggl\{f(\phi)R - \omega(\phi)\frac{\phi_{;\rho}\phi^{;\rho}}{\phi}\biggr\}.$$

- It is not the most general (Horndeski is, from the point of view of leading to second order equations), but it covers a large sample of theories.

# Non-minimal coupling

## Conformal transformation

- Let us fix,

$$f(\phi) = \phi.$$

- Performing a conformal transformation,

$$g_{\mu\nu} = \phi^{-1}\tilde{g}_{\mu\nu},$$

and writing,

$$\frac{d\phi}{d\sigma} = \left|\frac{3 + 2\omega}{\phi^2}\right|^{\frac{1}{2}},$$

we end up with,

$$\mathcal{L} = \sqrt{-g}\left\{R - \epsilon\sigma_{;\rho}\sigma^{;\rho}\right\}, \quad \epsilon = \pm.$$

# Non-minimal coupling

## Solutions

- A class of static, spherically symmetric solutions are,

$$ds^2 = P^{-\xi}\left\{P^a dt^2 - P^{-a} du^2 - P^{1-a} u^2 d\Omega^2\right\},$$

with

$$P(u) = 1 - 2\frac{k}{u} \quad , \quad \phi = P^{\xi},$$

$$\frac{b}{k} = a \quad , \quad a^2 = 1 - (3 + 2\omega)\xi^2,$$

$$2k^2 \mathrm{sign} k = 2b^2 + \epsilon C^2.$$

Black holes are only possible when $3 + 2\omega < 0$, corresponding to the phantom configuration in the Einstein's frame.

# Black holes with hair

## General features

- The solutions described before, in the minimal and non-minimal coupling, are black holes?
- They have striking features:
  1. Their surface gravity is zero. It means that their Hawking temperature (if it is possible to define it!) is zero. For this reason, these black holes has been called **Cold black holes**.
  2. The surface of the event horizon is infinite. This means an infinite entropy, if we follow the area law? But, if the temperature is zero, how the entropy can be infinite?
  3. The tidal forces are infinite over the event horizon hypersurface. But, point particles can cross the hypersurface.
  4. The solutions are asymptotically flat. Hence, static observers at the infinity can be defined.

# Black holes with hair

## The hair

- There is another important feature:
  The solutions contain a scalar charge, hence these solutions contain hair.
- This happens even in the Einstein's frame, if the energy conditions are violated.
- This fact points to the limitations of the so-called *no-hair theorem*.
- In fact, this theorem seems to be restricted to very specific situations.

# Black holes in the minimally coupled models

## No black hole for ordinary scalar field

- In fact, the general solution for these scalar black holes reveals two types of asymptotically flat black hole, besides the trivial (Schwarzschild) one.
  1. Type B1 black hole, where the horizon can be crossed in a finite proper time by an in falling particle.
  2. Type B2 black hole, where the horizon is reached in a infinite proper time for an in falling particle.
- In both cases, there is always a throat at a finite distance of the horizon.
- K.A. Bronnikov, G. Clement, C.P. Constantinidis, J.C. Fabris, Grav.&Cosmol. **4**, 128(1998); K.A. Bronnikov, M.S. Chernakova, J.C. Fabris, N. Pinto-Neto, M.E. Rodrigues, IJMPD**17**, 25(2008).

# Stability

## The GR case

- The usual GR black hole solutions are stable.
- These Cold Black Holes, are they stable?
- Stability of black holes are a very delicate subject, technically and conceptually.
  - Technically because is almost impossible to integrate the perturbed equations even in the simplest case, that of pure radial perturbations.
  - Conceptually because the results may be different for different perturbation formalisms.

# Stability

## Radial perturbations

- Let us consider just radial perturbations:

$$\alpha = \alpha_0(r) + \delta\alpha(r,t) \quad , \qquad \beta = \beta_0(r) + \delta\beta(r,t),$$
$$\gamma = \gamma_0(r) \quad + \quad \delta\gamma(r,t),$$

where $\alpha_0$, $\beta_0$ and $\gamma_0$ are the previously found background solutions and $\delta\alpha$, $\delta\beta$ and $\delta\gamma$ are (time dependent!) perturbations around them.

# Stability

## Gauge condition

- In analysing perturbations around the Schwarzschild solution, for example, it is very convenient to choose the coordinate condition,

$$\delta\beta = 0.$$

- However, here the situation is more delicate.
- Since there is a throat in the solutions at a given distance of the horizon, the condition above comes out to be a *physical condition*: only perturbations that does not affect the throat are considered.

# Stability

## Master equation

- It comes out, for the Cold Black Holes, to choose a coordinate condition similar to that of the background.

$$\delta\alpha = \delta\gamma + 2\delta\beta.$$

- The perturbed equations may be reduced to a single master equation connected with the perturbation of the scalar field:

$$e^{4\beta(u)}\delta\ddot{\phi} - \delta\phi'' = 0.$$

- This solution can be reduced to

$$\delta\phi'' + e^{4\beta(u)}\omega^2\delta\phi = 0.$$

# Stability

## Stables

- The condition of stability is fixed as it follows:
  - Fix the boundary conditions at horizon and at infinity;
  - Verify if there is solutions connecting these boundary conditions with $\omega^2 < 0$;
  - If there are solutions satisfying the above conditions connecting those boundary conditions, hence there are divergent growing modes, and the solutions are unstable.
  - If not, the solutions are stable.
- Results:
  - Black hole solutions are stable;
  - Naked singularity solutions are unstable.

# Stability

## Gauge invariant approach

- J.A. Gonzalez, F.S. Guzman, O. Sarbach, Class. Quantum Grav. **26**, 015010 (2009).
- Let us consider now the scalar field and a given metric perturbations that transform under the radial reparametrizatioin $\Delta u$ as,

$$\begin{aligned} \delta\phi &\rightarrow \bar{\delta\phi} = \delta\phi + \phi'\Delta u, \\ \delta r &\rightarrow \bar{\delta r} = \delta r + r'\Delta u. \end{aligned}$$

- The combination,

$$\Psi = r'\delta\phi - \phi'\delta r,$$

is invariant under that coordinate transformation.

# Stability

## Gauge invariant approach

- The final equation is:

$$e^{2(\alpha-\gamma)}\delta\ddot{\phi} - \delta\phi'' - \delta\phi'(\gamma' + 2\beta' - \alpha') + U\delta\phi = 0,$$

$$U \equiv e^{2\alpha}\biggl\{\epsilon(V - e^{-2\beta})\frac{\phi'^2}{\beta'^2} + 2\frac{\phi'}{\beta'}V_\phi + \epsilon V_{\phi\phi}\biggr\},$$

where $V$ is a possible potential term.

# Stability

## Gauge invariant approach

- The singularity due to the term $\beta'$ in the denominator leads to singularities for the solutions with a throat.
- These singularity may be removed by further transformation, and a Schrödinger-type equation with a regular potential is obtained.
- Performing this analysis, all the previous solutions (which have throat) come out to be unstable, including those corresponding to black holes.
- K.A. Bronnikov, J.C. Fabris, A. Zhidenko, Eur.Phys.J. **C71**, 1791(2011).

# Stability

## The discrepancy

- The reason for this discrepancy is not clear.
- Perhaps, in the gauge $\delta\alpha = 2\delta\beta + \delta\gamma$, we end up with a master equation for a pure gauge mode.
- Or perhaps the singularity existing in the potential of the gauge invariant approach led to non-physical modes.

# Black holes in $K$-Essence Theories

## General framework

- Let us consider the Lagrangian of $K$-Essence model:

$$\mathcal{L} = \sqrt{-g}\biggl\{R - f(X)\biggr\},$$

where $f(X)$ is a general function of the kinetic term for the scalar field,

$$X = \phi_{;\rho}\phi^{;\rho}.$$

# Black holes in $K$-Essence Theories

## General framework

- For the static, spherically symmetric case, the equations of motion are:

$$\begin{aligned} -2\beta'' - 3\beta'^2 + 2\beta'\alpha' + e^{2(\alpha-\beta)} &= e^{2\alpha}\left\{\frac{1}{2}f(X) + V(\phi)\right\}, \\ 2\gamma'\beta' + \beta'^2 - e^{2(\alpha-\beta)} &= e^{2\alpha}\left\{Xf_X - \frac{1}{2}f(X) - V(\phi)\right\}, \\ 2\gamma'\beta' + \beta'^2 - e^{2(\alpha-\beta)} &= -e^{2\alpha}\left\{\frac{1}{2}f(X) + V(\phi)\right\}, \\ \left[f_X e^{2\beta+\gamma-\alpha}\phi'\right]' &= V_\phi. \end{aligned}$$

# Black holes in $K$-Essence Theories

## No-go theorem

- For $V = 0$, it is possible to show that it can exist only cold black holes, even in absence of exact solutions.
- To do so, the coordinate condition is chosen to be,

$$\alpha = -\gamma,$$

- The following definitions are use:

$$e^{2\gamma} = A, \quad e^{\beta} = r(u).$$

# Black holes in $K$-Essence Theories

## No-go theorem

- A combination of the equations leads to:

$$
\begin{aligned}
2A\frac{r''}{r} &= Xf_X,\\
A(r^2)'' - A''r^2 &= 2,\\
2\frac{r''}{r} &= -C^2\frac{f_X}{r^4},
\end{aligned}
$$

where $C$ is connected with the scalar charge.

# Black holes in $K$-Essence Theories

## No-go theorem

- A first integral can be obtained:

$$\left(\frac{A}{r^2}\right)' = \frac{6m - 2u}{r^4}.$$

# Black holes in $K$-Essence Theories

## No-go theorem

- A horizon implies,

$$A \to 0.$$

- But, there is the relation,

$$Af_X\phi' = \frac{C}{r^2}.$$

- If the components of the energy momentum tensor must remain finite, and since they contain $f_X$ and $\phi'$, in order to avoid singularities, a non singular horizon can only happens at

$$r \to \infty$$

- Hence, the black hole must be cold.

# Black holes in $K$-Essence Theories

## A particular solution

- Let us consider,

$$f(X) = X^n.$$

- The equations of motion are the following:

$$-2\beta'' - 3\beta'^2 + 2\beta'\alpha' + e^{2(\alpha-\beta)} = \frac{\epsilon}{2}e^{2(1-n)\alpha}\phi'^{2n} + e^{2\alpha}V$$

$$2\gamma'\beta' + \beta'^2 - e^{2(\alpha-\beta)} = (2n-1)\frac{\epsilon}{2}e^{2(1-n)\alpha}\phi'^{2n} - e^{2\alpha}V$$

$$\beta'' + \gamma'' + \beta'(\gamma' + \beta' - \alpha') + \gamma'^2 - \alpha'\gamma' = -\frac{\epsilon}{2}e^{2(1-n)\alpha}\phi^{2n} - e^{2\alpha}V$$

$$(2n-1)\phi'' + [\gamma' + 2\beta' + (1-2n)\alpha']\phi' = \frac{\epsilon}{n}e^{-2(1-n)\alpha}\phi'^{2(1-n)}V_\phi$$

# Black holes in $K$-Essence Theories

## A particular solution

- Solutions for $n = 1/2$, $V = \Lambda =$ constant, and using the coordinate condition $\alpha = \gamma + 2\beta$:

$$
\begin{aligned}
ds^2 &= \frac{\Lambda^2}{\cosh^4\theta}dt^2 - du^2 - \frac{\cosh^2\theta}{\Lambda^2}d\Omega^2, \\
\phi &= \frac{2}{3}\epsilon\Lambda(-3\theta + 4\tanh\theta) + \phi_0, \\
\theta &= \frac{\Lambda}{3}u.
\end{aligned}
$$

# Black holes in $K$-Essence Theories

## A particular solution

- in the quasi-global coordinate system ($\alpha = -\gamma$):

$$ds^2 = (1-\Lambda\rho^2)^2 dt^2 - \frac{d\rho^2}{(1-\Lambda\rho^2)^2} - \frac{d\Omega^2}{(1-\Lambda\rho^2)}.$$

- A cold black hole (?) with a very curious structure.
- It is a kind of type B2 cold black hole. But, instead of having a flat asymptotic connect with a horizon, we have two horizons at $\rho = \pm 1/\sqrt{\Lambda}$, corresponding to $u \to \pm\infty$.

# Conclusions

- Scalar field allows to obtain exact black hole solution, but only if the energy conditions are violated.
- This remains true even in some other context, like the string-inspired dilatonic action (works of Glauber Tadiesky Marques, Manuel E. Rodrigues, in collaboration with G. Clément).
- These black holes have have zero gravity surface, infinite event horizon surface, infinite tidal forces.
- Their stability is controversial.
- $K$-Essence models give origin a similar structure, but with some other exotic configurations, to be still better exploited.