New solutions of charged regular black holes and their stability

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Introduction to regular black holes

• Regular black holes: Black holes without spacetime singularities.
• Spacetime inside black holes is regular.

• Singularity theorems
  
  Cosmological context (Hawking 1967, Hawking 1967)
  Gravitational collapse (Penrose 1965, Hawking & Penrose 1970)
  Predict a spacetime singularity inside a black hole.
They do not claim the singularities are inevitable in general condition.

Example:

Singularity theorems

Strong energy condition: \( R_{\mu \nu} l^\mu l^\nu \geq 0 \) for non-spacelike \( l^\mu \)

(Perfect fluid: \( \rho + p \geq 0, \rho + \Sigma p \geq 0 \))

de Sitter \( \Rightarrow R_{\mu \nu} l^\mu l^\nu = \Lambda g_{\mu \nu} l^\mu l^\nu \leq 0 \)

Regular black holes related to \( \Lambda ? \)
Types of regular black holes

• Four types of regular black holes (Bronnikov et al. 2007)

Containing centres
• Restrict to spherically symmetric case.

• Two kinds of regular black hole models.

• Common properties: two horizons and a de Sitter (like) core
(i) Bardeen type

The spacetime is described by a sufficiently smooth metric.

ex) Bardeen metric

\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) , \]

\[ f(r) = 1 - \frac{2mr^2}{(r^2 + e^2)^{3/2}} \]

(Bardeen 1968, taken from Ansoldi 2008)

m : black hole mass, e: black hole charge

• r → 0 (de Sitter like spacetime)

\[ f(r) \rightarrow 1 - \frac{2mr^2}{e^3} \quad \left( f(r) = 1 - \frac{r^2}{L^2} \right) \]

\[ L = \sqrt{\frac{3}{\Lambda}} \]

• Many studies of this type

(Dymnikova 1992, Bronnikov 2001, Dyminikova & Galaktionov 2005, etc.)
(ii) Two different spacetimes are matched by a thin shell or a surface.

\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \]

Typically, the interior of the shell is de Sitter spacetime. (e.g. Frolov et al. 1990, Lemos & Zanchin 2011)

\[ f(r) = \begin{cases} 
    f_{\text{dS}}(r) \equiv 1 - \frac{r^2}{L^2}, & \text{for } r < R(\tau), \\
    \text{black hole spacetime}, & \text{for } r > R(\tau), 
\end{cases} \]

L : de Sitter horizon radius, R : radius of the shell

We focus on this type of regular black holes.
The studies so far...

Exterior spacetime

Schwarzschild spacetime

Shell with mass and pressure (Frolov et al. 1990)

Stability analysis (Balbinot & Poisson 1991)

Depends on the equation of state of the shell.

Reissner-Nordstrom spacetime

Massless thin shell (Lemos & Zanchin 2011)

Stability ?

Massive shell ?

This work
• Why de Sitter spacetime?

**Conjecture** (Sakharov 1966, Gliner 1966, Markov 1982)

Spacetime curvature has an upper limit of the order of the Planck scale, $l_p$, due to the quantum effect.

Curvature invariant

$$\mathcal{R}^2 \equiv R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} \leq \frac{\alpha}{l_p^4}$$

(Schwarzshild black hole

$$\mathcal{R}^2 = \frac{48m^2}{r^6}$$

Collapsing matters turn into a de Sitter phase at the limit.

$$\left( \mathcal{R}^2 = \frac{24}{L^4} \right)$$
Physically reasonable?

(Previous studies assume the conjecture. Frolov et al. 1990, Balbinot & Poisson 1991)

This work ➔ assume the conjecture but do not verify it.

Related studies:
Collapsing of charged shells (Chase 1970, Boulware 1973)
• Charged spherically symmetric black holes
  (Reissner-Nordstrom black holes)

Metric

\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta \, d\phi^2), \quad f(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} \]

m : black hole mass, Q : black hole charge
Horizons exist when \( m \geq Q \).
Event horizon \( r_+ \), inner horizon \( r_- \), where \( f(r) = (r - r_+)(r - r_-)/r^2 \).
Penrose diagram ➔
(Spacetime diagram)

Spacetime singularity at $r=0$.

Constant $r$

$$ds^2=-(r - r_+) (r - r_-)/r^2 \, dt^2$$

$r < r_-, r > r_+$:

timelike hypersurface

$r_- < r < r_+$:

spacelike hypersurface

Spatial (radial)

$i^0 : \text{finite } t, r \to \infty$

$i^\pm : t \to \pm \infty, \text{finite } r$
Replace the singularity (at \( r=0 \)) by de Sitter spacetime and a thin shell.

- Locate a shell at constant \( r \).
- Two possible regular black hole models.
- Two horizons are needed.
(1)

de Sitter 
+ 
spacelike thin shell
+ 
Reissner-Nordstrom

$L < R < r_+$

$L$: de Sitter horizon radius
$R$: shell’s radius

Stability of a spacelike surface?

(Uncharged case; Frolov et al. 1990, Balbinot & Poisson 1991)
We consider this model.

(Cauchy horizon instability?)
Our aims

• To find new solutions of charged spherically symmetric regular black holes having a massive thin shell.

• To examine their stability against the radial displacement of the shell.

• To examine the stability in the massless limit of the shell.
Method of analysis

• To find stationary solutions: Junction condition of spacetimes.
• To analyse their stability: To analyse the stability of the shell \( \Rightarrow \) equation of motion of the shell

These can be obtained from the Israel junction condition.
Junction conditions

- Two different spacetimes, $V^+$ and $V^-$, are matched by a thin shell.
- $h_{ab}$: induced metric $\Rightarrow [h_{ab}] = 0$
- $K_{ab}$: extrinsic curvature
- $S_{ab}$: energy momentum tensor of the shell

\[ 8\pi S_{ab} = \epsilon ( [K_{ab}] - h_{ab}[K] ), \text{ where } [K_{ab}] = K_{ab}^+ - K_{ab}^- \]

- $V^+$: Reissner-Nordstrom, $V^-$: de Sitter
- Metric

\[
d s^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),
\]

\[
f(r) = \begin{cases} 
    f_{ds}(r) & \equiv 1 - \frac{r^2}{L^2}, \quad \text{for } r < R(\tau), \\
    f_{RN}(r) & \equiv 1 - \frac{2m}{r} + \frac{Q^2}{r^2}, \quad \text{for } r > R(\tau),
\end{cases}
\]

\(m\) : black hole mass, \(Q\) : black hole charge , \(\tau\) : proper time of the shell

- Assumptions

The shell is composed of dust.

The shell exists inside the de Sitter horizon \(L\) and the inner horizon of the black hole metric \(r^-\). (\(R < r^-\) & \(R < L\))

\(\Rightarrow\) The shell is timelike.
Equation of motion of the shell

• Energy conservation $S_{b;a}^a = 0$
  $\Rightarrow$ Proper mass of the shell $M$ is conserved.

• Equation of motion of the shell

\[
\sqrt{\dot{R}^2 + 1 - \frac{R^2}{L^2}} - \sqrt{\dot{R}^2 + 1 - \frac{2m}{R} + \frac{Q^2}{R^2}} = \frac{M}{R}
\]

More useful form:

\[
\dot{R}^2 + V(R) = -1, \quad V(R) = -\left(\frac{R^3}{L^2} + \frac{Q^2}{R} - \frac{2m}{2M} - \frac{M}{2R}\right)^2 - \frac{R^2}{L^2}
\]
Solve \( V(R) = -1 \) and \( dV(R)/dR = 0 \) with fixed \( m \) and \( M \).

Stable (unstable) solutions have

\[ d^2V(R)/dR^2 > 0 \quad \text{(stable)} \]
\[ d^2V(R)/dR^2 < 0 \quad \text{(unstable)} \]

Ex)
Results and discussions

• Stationary solutions (de Sitter horizon radius $L=1$)
  We admit the solutions with negative mass of the shell.

Sequences of the solutions with a fixed black hole mass $m$.

- Minimum $|M|=10^{-4}$
- Stable solutions
  $0.87 \ L < m < 2 \ L,$
  $0.86 \ L < R < L$
  $0 < M < 0.4$

(Uchikata, Yoshida & Futamase 2012)

Stable (unstable) solutions have positive (negative) mass of the shell.
Sequences of the solutions with a fixed black hole mass $m$.

- Stable solutions: $0.87 \, L < m < 2 \, L$, $0.88 < Q/m < 1$

(Uchikata, Yoshida & Futamase 2012)
Massless limit of the shell

• The same regular black hole model but with a massless shell. (Lemos & Zanchin 2011)
• Stability of the solutions is unknown.
• No equation of motion for M=0.

\[ \sqrt{\dot{R}^2 + 1 - \frac{R^2}{L^2}} - \sqrt{\dot{R}^2 + 1 - \frac{2m}{R} + \frac{Q^2}{R^2}} = \frac{M}{R} \]

• Consider M→0 limit (|M|=10^{-4}) of the present results.
• In the massless limit of the shell, $M \to 0$, we obtain

\[
\begin{align*}
R^2 &= \frac{QL}{\sqrt{3}} + \frac{1}{3}ALM + O(M^2), \\
m &= 2\frac{R^3}{L^2} + BM + O(M^2),
\end{align*}
\]

\[
\frac{d^2V}{dR^2} \approx \frac{12R_0\sqrt{L^2 - R_0^2}}{ML^3},
\]

Solutions with massless shell small $M$ correction terms
(Lemos & Zanchin 2011)

• Our numerical solutions of $|M| = 10^{-4}$ satisfy above relations.
• We have both stable and unstable solutions in $M \to 0$ limit.
Physical scales of the regular black holes

- Stable solutions: black hole mass $m \sim L$
- Estimation of $L$
  (Planck length $l_p$, the GUT scale, electroweak scale)

Based on the conjecture (curvature has an upper limit)

  $\Rightarrow$ previous studies approximate as $L \sim l_p$.
  (Frolov et al. 1990, Balbinot & Poisson 1991)

Black hole mass $\Rightarrow$ that of primordial black holes ($< 10^{10} \text{kg}$)
Summary

• We have constructed a charged regular black hole with a massive thin shell.
• We have obtained stationary solutions. The black hole mass of the stable solutions tend to have $m \sim L$.
• In the massless limit of the shell, the solutions approach the massless shell case. And they may be stable.
• Stable regular black holes are of the size of primordial black holes.

Future work
Extend the analysis to the rotating case.