Structural design optimization of wind turbine towers

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Abstract

This paper describes several optimization models for the design of a typical wind turbine tower structure. The main tower body is considered to be built from uniform segments where the effective design variables are chosen to be the cross-sectional area, radius of gyration and height of each segment. The nacelle/rotor combination is regarded as a rigid non-rotating mass attached at the top of the tower. Five optimization strategies are developed and tested. The last one concerning reduction of vibration level by direct maximization of the system natural frequencies works very well and has shown excellent results for both tower alone and the combined tower/rotor model. Extensive computer experimentation has shown that global optimality is attainable from the proposed discretized model and a new mathematical concept is given for the exact placement of the system frequencies. The normal mode method is applied to obtain forced response for different types of excitations. The optimization problem is formulated as a nonlinear mathematical programming problem solved by the interior penalty function technique. Finally, the model is applied to the design of a 100-kW horizontal axis wind turbine (ERDA-NASA MOD-0). It has succeeded in arriving at the optimum solutions showing significant improvements in the overall system performance as compared with a reference or baseline design.

Keywords: Renewable energy; Wind turbine; Vibration; Tower structures; Dynamics; Optimization

1. Introduction

Many wind energy research and development programs have been initiated since the 1973 oil crisis, and different configurations of wind turbines have been installed in many countries. These clean energy sources can make a substantial and economically competitive contribution to the future energy needs. Concerning system design optimization, most of the published research work has mainly been focused on power-output economics and cost optimization [1,2]. Little may be found that deals with the problem of structural design optimization. It is the main intent of the present study to consider the basic aspects of design optimization of the combined tower/rotor structure of a wind turbine. A good survey of the various theories and numerical methods of structural design optimization can be found in Refs. [3,4] which are usually classified into two groups: the mathematical programming approach, and the optimality criteria approach. The latter was employed by Takewaki [5] who considered optimization of a concrete tower structure with round tubular cross section. An approximate concept was applied for finding the optimal bending stiffness distribution which minimizes the total structural weight for a specified
fundamental natural frequency. Later on, Takewaki solved the same problem taking into consideration the flexibility effect of the tower base connection [6]. He used finite element method and a piecewise-linear function to approximate the stiffness distribution along the tower height.

In the present paper, several optimization strategies for the design of a wind turbine tower are developed and tested, and different forms of the objective function are examined. The tower is considered to be built from uniform round tubular segments (modules), where the effective design variables are the cross-sectional area, radius of gyration and height of each segment. Isolated tower dynamics including the complete kinematical analysis and formulation of the applied loads are treated in detail. A simplified set of the governing dynamical equations of motion is given in an appropriate non-dimensional form, and an exact method for the determination of the dynamic characteristics and forced response is presented. The final model formulation identifies the optimum values of the design variables which make the tower structure fulfil its design objectives in the best possible manner while satisfying all design constraints. The problem is formulated as a nonlinear mathematical programming problem. Optimum design solutions are obtained by the interior penalty function method coupled with Powell’s multi-dimensional and quadratic-interpolation one-dimensional search techniques.

Implementation of the final model form to an existing wind turbine; the ERDA-NASA MOD-0 machine, is given and the optimum design trends are examined and discussed. It is demonstrated that global optimality can be attained from the proposed discretized models, and exact placement of the tower frequencies is also achievable.

2. Optimization problem formulation

In formulating an optimization problem, three principal phases must be considered:

- Definition and measure of the system design objectives.
- Selection of the design variables and preassigned parameters.
- Definition of the design constraints.

2.1. Basic assumptions

- The basic structural model of the tower is represented by an equivalent long, slender cantilever beam built from segments (modules) having different but uniform cross-sectional properties. The tower is cantilevered to the ground, and is carrying a concentrated mass at its free end approximating the inertia properties of the nacelle/rotor unit. This mass is assumed to be rigidly attached to the tower top.
- Material of construction is linearly elastic, isotropic and homogeneous. The tower has a thin-walled circular cross-section.
- The Euler–Bernoulli beam theory is used for predicting deflections. Secondary effects such as axial and shear deformations, and rotary inertia are neglected.
- Distributed aerodynamic loads are restricted to profile drag forces. A two-dimensional (2D) steady flow model is assumed.
- Nonstructural mass will not be optimized in the design process. Its distribution along the tower height will be taken equal to some fraction of the structural mass distribution.
- Structural analysis is confined only to the case of flapping motion (i.e. bending perpendicular to the plane of rotor disk).

2.2. Tower design objectives

A wind turbine tower is the main structure which supports rotor, power transmission and control systems, and elevates the rotating blades above the earth boundary layer. A successful structural design of the tower should ensure efficient, safe and economic design of the whole wind turbine system. It should provide easy access for maintenance of the rotor components and sub-components, and easy transportation and erection. Good designs ought to incorporate aesthetic features of the overall machine shape.

In fact, there are no simple criteria for measuring the above set of objectives. However, it should be recognized that the success of tower structural design is judged by the extent to which the wind turbine main function is achieved. The different optimization strategies considered herein are discussed below.

2.2.1. Light weight design

A minimum weight structural design is of paramount importance for successful and economic operation of a wind turbine. The reduction in structural weight is advantageous from the production and cost points of view. For a tower composed of \( N_s \) segments the weight (mass) function, to be minimized, can be expressed in the non-dimensional form

\[
\min_{k=1}^{N_s} D_k t_k H_k
\]

For the definition of the various parameters, refer to Table 1 and Table 2.
2.2.2. High stiffness

The main tower structure must possess an adequate stiffness level. Maximization of the stiffness is essential to enhance the overall structural stability and decrease the possibility of fatigue failure. For a cantilevered tower, stiffness can reasonably be measured by the magnitude of a horizontal force applied at the free end and producing a maximum deflection of unity. This can be expressed mathematically as in Ref. [7]:

\[
\text{maximize; } S = 1/ \sum_{k=1}^{N} \frac{H_k}{I_k} \left[ 1 - (x_{k+1} + x_k) \right] + \frac{1}{2} \left( x_{k+1}^2 + x_k x_{k+1} + x_k^2 \right)
\]  

(2)

2.2.3. High (stiffness/mass)-ratio

Perhaps, maximization of the stiffness-to-mass ratio, which is directly related to the physical realities of the design, is a better and more straightforward design criterion than maximization of the stiffness alone or minimization of the structural mass alone. The related mathematical expression can be obtained by dividing Eq. (2) by Eq. (1).

2.2.4. Design for minimum vibration

Minimization of the overall vibration level is one of the most cost-effective solutions for a successful wind turbine design. It fosters other important design goals, such as long fatigue life, high stability and low noise level. Two different criteria for measuring vibration reduction are stated in the following sections [7].

2.2.4.1. Frequency-placement criterion. Reduction of vibration can be achieved by separating the natural frequencies of the structure from the exciting frequencies to avoid large amplitudes caused by resonance [8]. This may be measured by minimizing the performance index

<table>
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<td>Parameter</td>
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<td>Thickness</td>
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<td>Cross-sectional area (structurally effective)</td>
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<tr>
<td>Second moment of inertia of the cross section</td>
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<td>Radius of gyration</td>
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<tr>
<td>Mass density</td>
<td>(\rho_0)</td>
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<tr>
<td>Non-structural mass per unit length</td>
<td>(m_{ns0})</td>
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<tr>
<td>Structural mass/length</td>
<td>(m_\rho)</td>
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<tr>
<td>Mass/unit length</td>
<td>(m_0)</td>
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<tr>
<td>Young’s modulus</td>
<td>(E)</td>
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Thin - Walled Round Tubular Configuration
\[
\min \sum_{i} W_i (\omega_i - \omega_i^*)^2
\]  \( (3) \)

where \( W_i \) are weighting factors, \( \omega_i \) are the actual frequencies and \( \omega_i^* \) are the corresponding desired (target) values chosen to be within close ranges (called frequency-windows) of those of a reference or baseline tower design, which are well separated from the critical exciting frequencies.

2.2.4.2. Maximum frequency criterion. Another alternative for reducing vibrations is the direct maximization of the system natural frequencies. Higher natural frequencies are favorable for reducing both of the steady-state and transient responses of the tower. Szyszkowski [9] considered maximization of the fundamental frequency for given structural weight via a multi-modal optimality criterion. Another work dealing with optimum shapes of beams for maximum fundamental frequency has been studied by Masad [10], who presented an efficient numerical method for computing the rate of change of the associated eigenvalues. The present work considers a different optimization strategy by maximizing a weighted-sum of the system natural frequencies. This is described mathematically by the following linear composite form of the objective function [7]:

\[
\max \sum W_i F_{bi}
\]  \( (4) \)

where \( F_{bi} = \sqrt{\omega_i} \) are the non-dimensional frequency parameters of bending vibrations (Eq. (14)).

2.3. The preassigned parameters

The following tower variables will be given preassigned fixed values in order to decrease the dimensionality of the optimization problem:

1. tubular single-pole configuration
2. total height, \( H \)
3. type of cross section is thin-walled hollow circular cross section
4. material of construction is chosen to be steel

<table>
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<td>Area</td>
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<td>Mass moment of inertia</td>
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<tr>
<td>Natural frequency</td>
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<td>Damping factor</td>
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* E.g. Dim. \((F_z) = \text{Non-dim.}(F_z) \times (EI_0/H_0^2) \text{ or } F_x \leftarrow F_x \times (EI_0/H_0^2)\).
2.4. Design variables

The design variables which are subject to change in the optimization process are chosen to be the radius of gyration, cross-sectional area and height of each module composing the main tower structure. For a round thin-walled tubular configuration, these correspond to the mean diameter, wall-thickness and height of each module. In other words, the design-variable vector $\mathbf{x}_d$ is defined as

$$\mathbf{x}_d = \{(D_1, t_1, H_1), (D_2, t_2, H_2), \ldots, (D_k, t_k, H_k), \ldots\},$$

where

$$k = 1, 2, \ldots, N_s,$$

which is of order $(3N_s \times 1)$.

2.5. Design constraints

For the present model formulation, design constraints are stated in the following section.

2.5.1. Behavioral constraints

Strength requirements

$$\frac{\sigma_{ek}}{\sigma_{al}} - 1.0 \leq 0.0, \quad k = 1, 2, \ldots, N_s$$

Maximum deflection

$$\frac{W_{\text{max}}}{W_{\text{al}}} - 1.0 \leq 0.0$$

Resonance avoidance

$$\left(1.0 - \Delta_i \leq \frac{\omega_i}{\omega_{i1}} \leq 1.0 + \Delta_i\right)$$

Mass limitation

$$(M/M_{al}) - 1.0 \leq 0.0$$

The various parameters and symbols of the above inequalities are defined in the following:

- $\sigma_{ek}$: maximum effective axial stress within the $k$th module
- $\sigma_{al}$: allowable tensile or compressive stress
- $W_{\text{max}}$: maximum tip deflection
- $W_{\text{al}}$: an allowable value of deflection
- $\omega_i$: target (or desired) frequencies that are well separated from the exciting ones
- $\Delta_i$: allowable tolerances of frequencies
- $M_{al}$: allowable upper limiting value of the structural mass

Regarding buckling stresses, the present simplified analysis only imposes lower bounds on the wall thickness along the tower height in order to avoid local instability.

2.5.2. Side constraints

$$D_{kl} \leq D_k \leq D_{ku}$$

$$t_{kl} \leq t_k \leq t_{ku}, \quad k = 1, 2, \ldots, N_s$$

where $D_{kl}$ and $D_{ku}$ are the lower and upper limiting values of the $k$th diameter respectively, and $t_{kl}$ is the lower bounding value determined either from the minimum available sheet thicknesses or from considerations of local wall instability. The upper limiting value $t_{ku}$ is imposed in order that the thickness $t_k$ may not violate the assumption of thin-walled tubular configuration (e.g. $t_{ku} = 0.1 D_k$). The total tower height is assumed to be equal to that of a reference design, i.e., $\sum H_k = 1$. In addition, the non-negativity constraints $H_k \geq 0$ must also be included to avoid having unrealistic negative heights.

2.6. Choice of the final model form: functional behavior of the objective function

Perhaps the most important part of the mathematical model is the objective function. Care must be exercised in constructing such a function because the design recommended by the model will directly depend on its form. In this section, we shall attempt to test the different alternatives of the objective function through comprehensive computer implementation.

2.6.1. Mass and stiffness optimization analysis

Using Eqs. 1 and 2, the level curves of the mass, stiffness and stiffness-to-mass ratio can be generated and plotted. Fig. 1 shows those of the stiffness/mass ratio of a two-segment cantilever model. In order to minimize the mass alone below its reference value of one the stiffness is substantially degraded as it sinks to its lowest levels. The opposite trend is true, in order to raise the stiffness level above the reference value of 3.0, unacceptable high values for the mass can be reached and, hence, unrealistic cost of fabrication and production. Examining the well-behaved stiffness/mass function, it is seen that the middle region of the design space encompasses the global optimum solutions where balanced improvement in both mass and stiffness is attained. Several other configurations including two, three and four segments have been implemented and tested. It has been confirmed that maximization of the stiffness-to-mass ratio is a much better and straightforward measure of the system performance. Minimization of the mass alone or maximization of stiffness alone requires imposition of lower limits on stiffness or
upper limits on mass, respectively. The choice of the values of such limits is not easy.

2.6.2. Exact frequency optimization analysis

Vibration reduction may be achieved either by separating the natural frequencies from the exciting ones or by directly maximizing these frequencies. Whatever is the approach taken, it is useful to begin with studying the behavior of the system natural frequencies and see how they are changed with the selected design variables.

2.6.2.1. Frequency behavior of tower alone: new mathematical concept. Consider the simplest case of a uniform one-segment cantilever beam with non-dimensional height $H_1$ and radius of gyration $r_1$. The associated transcendental equation of bending frequencies can be shown to have the form

$$\cos \left( \frac{\Gamma_b H_1}{\sqrt{r_1}} \right) \cosh \left( \frac{\Gamma_b H_1}{\sqrt{r_1}} \right) = -1 \quad (11)$$

Fig. 2 shows variation of the fundamental frequency parameter $\Gamma_b = \sqrt{\omega_1}$ with $r_1 = D_1$ and $H_1 = 1$ for the round thin-walled tubular configuration. The mass-to-thickness ratio, $M/t_1$, is also shown in the figure. It is seen that both of the frequency and mass functions are well behaved and continuous in the variable $D_1$. Now it is possible to choose prescribed values for $\Gamma_b$ and either one of the variables $r_1(=D_1)$ or $H_1$ and solve for the remaining variable. Extensive computer analysis for numerical frequency solutions has proved the validity of such a mathematical concept. It is thus possible to freely place the natural frequencies at any desired values, but of course within prescribed mass limitations. This useful conclusion has been confirmed by consideration of tower models built of more than one segment [7]. Typical results showing the associated mass and frequency isomert curves are depicted in Fig. 3 and Fig. 4.

2.6.2.2. Frequency optimization of the combined tower/rotor structure. Two major factors should be included when considering frequency analysis of a wind turbine tower [11]. The first factor is the tip mass approximating the inertia and mass properties of the nacelle/rotor unit at the top of the tower. The computed results for a selected case is shown in Fig. 5. It is remarked that the frequency behaves in a manner different from that of the case of the tower alone because the tip mass has considerable contribution to the total system mass. The second factor is the axial compressive gravity forces which may destabilize the tower in case of large wind generator systems. Fig. 6 shows variation of the

![Fig. 1. Stiffness/mass isomert curves for a two-segment tower model ($H_1 = H_2 = 0.5$, $D_1 = t_1 = 1.0$).](image-url)
fundamental frequency with the non-dimensional gravity parameter, $g$, for different values of the tip mass, $M_R$. Variation of slenderness ratio, $H_0/r_0$, is also shown in the figure for different tower heights. It is seen that a quick reduction in the frequency occurs when the axial compressive stress becomes close to the critical buckling stress of the tower. The calculated numerical values of the corresponding critical gravity factor, $g_{cr}$, are approximately equal to 1.1, 2.0, 3.35, and 7.8 for $M_R = 2.0, 1.0, 0.5$, and 0.0, respectively.

2.6.3. Final model form

Different formulations of the optimization model have been tested on the computer using the actual data of the ERDA-NASA-MOD-0 wind turbine [7]. It was found that the most significant and promising optimization strategy is the direct maximization of the system natural frequencies (Eq. (4)). Frequency-windows constraints (Eq. (8)) were discarded from the model formulation because of the difficulty in finding a feasible starting point in the selected design space. If it happened that one of the critical forcing frequencies were close to any one of the natural frequencies of the optimized tower structure, another value for the natural frequency can be chosen near its optimum value and the associated eigenvalue problem is resolved for any one of the design variables instead. This is one of the advantages of the present mathematical formulation.

In addition, computer experimentation has shown that minimization of the frequency-placement index of Eq. (3) results in a more time consuming optimization process and a slow rate of convergence towards the optimum solution. Besides, the attained optimum solutions were found to be strongly dependent on the chosen values of the target frequencies, which are rather arbitrarily chosen. Finally, it was decided to discard the constraints on the module's diameters because it was found that optimization favors increasing diameters for maximizing frequencies.

2.6.4. Determination of the weighting factors

There are many procedures reported in the literature that help extract the values of the weighting factors, $W_{fi}$ of Eq. (4). For instance, one may give the fundamental frequency the largest weight, while ignoring the importance of other higher frequencies. A more reliable procedure of adjusting the values of $W_{fi}$ is given in Ref. [7] which is summarized as follows:

- Initialize $W_{fi}$ by the reciprocal of the values of the frequencies of a reference uniform tower,

$$W_{fi} = 1/(\Gamma_{fi})_{\text{reference}}$$

- Normalize the resulting values by dividing each one
Fig. 3. Mass and frequency level curves of two-segment towers. (a) $\Gamma_{31}$-contours, height vs diameter ($D_1 = 1$, $t_1 = 1$, $t_2 = 0.5$).
(b) Mass contours, height vs diameter ($D_1 = 1$, $t_1 = 1$, $t_2 = 0.5$). (c) $\Gamma_{52}$-contours, thickness vs height ($D_1 = 1$, $D_2 = 0.5$, $t_1 = 1$).
by their sum in order to make the final sum equal to one, i.e.

\[ W_{fi} = W_{f}/ \sum W_{f} \]

Therefore, the final appropriate values of the weighting factors are given as:
- Tower alone: 61.0, 24.4, and 14.6%.
- Combined tower/rotor: 61.2, 23.6, and 15.2%.

3. Tower analysis

This section describes briefly the method of analysis of the bending motion of an isolated tower/rotor structural model. Fig. 7 shows the intermediate coordinates and degrees of freedom required to define motion. The associated transformation matrices and the complete kinematical analyses can be found in reference [7] where the more general case of bending-bending-torsional motion is given in detail. Distributed-load vectors along tower height are expressed in the undeformed inertial coordinate system \((x, y, z)\) and include aerodynamic, inertial, gravitational as well as damping contributions. The final approximate governing partial differential equation for the bending displacement \(W\) is cast in the following non-dimensional form:

\[
(W_{xx})_{xx} - (F_x W_x)_{x} + P_z = 0
\]

which must be satisfied over the tower height, i.e. \(0 \leq x \leq 1\). The non-dimensional equivalent distributed load \(P_z\) can be calculated from [7]:

\[
P_z = -m_T w_l - g_z w_l + P_{\text{aero}} + F_{2NB} \delta(x - 1) + M_{\text{YNB}} \eta(x - 1)
\]

where the concentrated tip force \(F_{2NB}\) and moment \(M_{\text{YNB}}\) transmitted from the \(N_B\)-rotating blades are transformed to distributed loads by the use of the Dirac-delta, \(\delta\), and the unit doublet, \(\eta\), functions. The distributed aerodynamic force is given by [12]

\[
P_{\text{aero}} = 4 \rho_{\text{air}} V_0^2 C_D \frac{\pi}{2} (H_0/D_0)f(t)
\]

where \(f(t)\) is a time-dependent function accounts for the dynamic and gusty nature of the wind, \(x_0\) is called the wind shear exponent, \(C_D\) is the drag coefficient and
\( V_0 \) is the wind speed at hub height \( H_0 \) non-dimensionalized by the term \( \sqrt{(E/\rho)} \). The associated boundary conditions are:

At \( x = 0 \) (cantilevered end)
\[
W_0 = 0 \quad \text{and} \quad W_x = 0.
\]

At \( x = 1 \) (free end)
\[
(-IW_{xx})_x + F_x W_x + M_R W_{tt} = 0
\]
\[
(-IW_{xx})_x = I_k(W_x)_{tt} = 0
\] (15)

3.1. Natural vibration analysis: the eigenvalue problem

The first step in the solution procedure is the determination of the free natural vibration characteristics by removing all of the forcing functions and considering only the homogeneous equation. The resulting eigenvalue problem is described by the 4th order ordinary differential equation:
\[
\gamma_i''''(\tilde{x}) - (f_{sk} \gamma_i'(\tilde{x}))' - \omega_k^2 \gamma_i(\tilde{x}) = 0
\] (16)

which must be satisfied over any module \( (k) \) extending from \( x = x_k \) to \( x = x_{k+1} \), or \( 0 \leq \tilde{x} \leq H_k \), where \( \tilde{x} = x - x_k \) is a local coordinate system. \( f_{sk} \) is a non-dimensional quantity representing compressive force due to gravity and is given by
\[
f_{sk} = g_k (x - x_k), \quad f_{i0} = \frac{1}{A_k} \left( M_R + \sum_{i=0}^{N} A_i H_i \right),
\]
\[
g_k = g/r_k^2 \quad \text{and} \quad \omega_k = \omega_1/r_k
\]

\( \omega_1 \) and \( \gamma_1 \) are the non-dimensional eigenvalues (natural frequencies) and eigenfunctions (mode shapes), respectively. For thin-hollow circular cross section, the non-dimensional properties are:
\[
A_k = D_k t_k, \quad I_k = D_k^3 t_k, \quad \text{and} \quad r_k = D_k.
\] (18)

The above differential equation has been shown to have an exact solution by the method of Frobenius in which the spatial functions \( \gamma_i(x) \) is represented by a four-term power series [7]:

![Fig. 4. Fundamental frequency for towers built of four segments.](image-url)
\[ \gamma_i(x) = \sum_{n=1}^{4} C_n \lambda_n(x), \]

\[ \lambda_n(x) = \sum_{m=n}^{\infty} a_{nm}(x)^{m-1}, \quad m \geq n \]

where \( \lambda_n \) are four linearly independent solutions and \( a_{nm} \) are the unknown coefficients given by

\[ a_{nm} = -\frac{g_k f_k a_{n,m-2}}{(m-1)(m-2)} + \frac{g_k (m-4) a_{n,m-3}}{(m-1)(m-2)(m-3)} + \frac{\omega_k^2 a_{n,m-4}}{(m-1)(m-2)(m-3)(m-4)} \quad m \geq n \]
Since an exact solution is available for one uniform segment, then a non-uniform tower structure built from \( N_s \) uniform segments must have an exact solution for its natural frequencies and mode shapes. The transfer matrix method has been found to be suitable for the present discretized model formulation. Application of the associated boundary conditions and consideration of the non-trivial solution yields the following exact frequency equation:

\[
\begin{align*}
\alpha_1 \omega_i^4 + \alpha_2 \omega_i^2 + \alpha_3 &= 0 \\
\end{align*}
\]

where the coefficients \( \alpha_i \) are defined as

\[
\begin{align*}
\alpha_1 &= I_R M_R (T_{23} T_{14} - T_{24} T_{13}), \\
\alpha_2 &= I_R (T_{24} T_{43} - T_{23} T_{44}) + M_R (T_{34} T_{13} - T_{33} T_{14}), \\
\alpha_3 &= (T_{33} T_{44} - T_{34} T_{43})
\end{align*}
\]

\( T_{ij} \) are the elements of an overall transfer matrix formed by taking the products of all the intermediate transfer matrices of the different segments. Detailed derivations can be found in Ref. [7]. Once the natural frequencies \( \omega_i, i = 1, 2, \ldots, N \), have been determined, the associated mode shapes \( \gamma_i(x) \) can be obtained from Eqs. (19) and (20).

### 3.2. Forced response

Application of the normal mode method to the governing equation of motion and the associated boundary conditions (Eq. (12)) results in \( N \)-independent set of uncoupled differential equations of the form

\[
\ddot{\xi}_i(t) + 2\eta_i \omega_i \dot{\xi}_i(t) + \omega_i^2 \xi_i(t) = Q_i/M_i, \\
i = 1, 2, \ldots, N
\]

where \( Q_i \) is called the \( i \)th generalized load given by

\[
\begin{align*}
Q_i(t) &= \int_0^1 \gamma_i(x) P_{aero}(x,t) \, dx + \gamma_i(1) F_{ZNB} \\
&\quad - \gamma_i(1) M_{iNB}
\end{align*}
\]

Assuming zero initial conditions, the response is given by:

\[
\ddot{\xi}_i(t) = (1/\omega_i^2) \int_0^t Q_i(\tau) \exp\left[ -\eta_i \omega_i (t - \tau) \right] \sin \omega_i (t - \tau) \, d\tau.
\]

where \( \omega_{id} = \omega_i \sqrt{1 - \eta_i^2} \) is the \( i \)th damped natural frequency and \( M_i \) is the \( i \)th generalized mass defined as:

\[
M_i = \int_0^1 m R^2 \gamma_i^2 \, dx + M_R \gamma_i^2(1) + I_R \gamma_i^2(1)
\]

### 3.3. Internal loads and stress analysis

After the response has been found, displacements,
velocities and accelerations can be substituted for the calculation of the applied distributed loads. The internal forces and moments can then be determined at any location $x$ along the tower height by integrating the equilibrium equations of motion. Stresses resulting from axial, bending, torsion and shearing effects can be systematically calculated at any point within the structure. For the present structural model of round tubular configuration the most significant stress components are $\sigma_{xx}$ and $\tau_{xy}$. Other components are usually neglected. Stresses are calculated within the limitations of the engineering beam theory of bending and torsion, and the combined stress is determined on the basis of Von Mises equivalent stress theory.

3.4. Other critical loading conditions

The most important critical loading conditions which should be taken into consideration in the design phase of the supporting tower structure were investigated in detail in Ref. [7]. They include the cases of suddenly applied severe wind, transient loads induced by tower base motion and vortex-shedding induced vibrations.

4. Optimization analysis by mathematical programming

The tower optimization model developed in the previous sections is a non-linear mathematical programming model. The model uses the interior penalty function technique [13] in finding the constrained optimum solution. In contrast to the exterior technique, the interior technique presents a series of improving feasible designs which gradually approach the final optimum design. For multi-dimensional search the model uses Powell’s technique. The quadratic interpolation method has been chosen for the determination of the
step size along the search direction. A computer program was developed to automate the tower analysis and design procedures. Detailed description of the program can be found in Ref. [7].

5. Model implementation: a case study

As a case study, the proposed optimization model was implemented on an experimental, horizontal axis wind turbine; namely, the ERDA-NASA MOD-0 [11]. Its rotor has two blades with 38.1 m diameter located downwind from the tower at 30 m height above the ground. The original tower configuration is a steel–lattice truss structure having four legs connected together by chord members.

5.1. The reference cantilever beam and baseline design data

Table 3 presents the pertinent data of the reference cantilever beam which has a round tubular cross-section type. Its total mass is chosen to be the same as that of the original MOD-0 configuration. Table 4 gives the associated values of the non-dimensional multiplier factors of the various quantities. The equivalent discretized model of the baseline tower design is determined on the basis of having the same total mass and stiffness and mass distributions as that of the actual MOD-0 design. It is selected to be composed of six segments with the non-dimensional design variables given in Table 5.

5.2. Results of structural analysis

Fig. 8 depicts the calculated frequencies and mode shapes of the reference cantilever beam and baseline designs. It is seen that the first and second frequencies of the baseline design are higher than those of the reference cantilever indicating a good tower structural configuration. The third frequency, however, is slightly less than its reference value. The fundamental frequency of the combined tower/rotor structure is about

<table>
<thead>
<tr>
<th>Table 3</th>
<th>Parameters of the reference cantilever beam (thin-walled round tubular configuration)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
<td>Notation (units)</td>
</tr>
<tr>
<td>Height</td>
<td>$H_0$ (m)</td>
</tr>
<tr>
<td>Mean diameter</td>
<td>$D_0$ (m)</td>
</tr>
<tr>
<td>Wall-thickness</td>
<td>$t_0$ (m)</td>
</tr>
<tr>
<td>Cross-sectional area</td>
<td>$A_0$ (m$^2$)</td>
</tr>
<tr>
<td>Second moment of inertia</td>
<td>$I_0$ (m$^4$)</td>
</tr>
<tr>
<td>Radius of gyration</td>
<td>$r_0$ (m)</td>
</tr>
<tr>
<td>Material of construction</td>
<td>Type: Steel</td>
</tr>
<tr>
<td>Mass density</td>
<td>$\rho_0$ (kg/m$^3$)</td>
</tr>
<tr>
<td>Young's modulus</td>
<td>$E$ (N/m$^2$)</td>
</tr>
<tr>
<td>Allowable normal stress</td>
<td>$\sigma_a$ (N/m$^2$)</td>
</tr>
<tr>
<td>Allowable shear stress</td>
<td>$\tau_a$ (N/m$^2$)</td>
</tr>
<tr>
<td>Poisson's ratio</td>
<td>$\nu$</td>
</tr>
<tr>
<td>Structural mass/length</td>
<td>$m_{0a}$ (kg/m)</td>
</tr>
<tr>
<td>Non-structural mass/length</td>
<td>$m_{0a}$ (kg/m)</td>
</tr>
<tr>
<td>Total mass/length</td>
<td>$m_0$ (kg/m)</td>
</tr>
<tr>
<td>Total structural mass</td>
<td>$M_{0a}$ (kg)</td>
</tr>
<tr>
<td>Total non-structural mass</td>
<td>$M_{0a}$ (kg)</td>
</tr>
<tr>
<td>Total mass</td>
<td>$M_0$ (kg)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 4</th>
<th>Non-dimensional multiplier factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-dimensional quantity</td>
<td>Notation</td>
</tr>
<tr>
<td>Concentrated moments</td>
<td>$M_x, M_y, M_z$</td>
</tr>
<tr>
<td>Concentrated forces</td>
<td>$F_x, F_y, F_z$</td>
</tr>
<tr>
<td>Distributed moments</td>
<td>$q_x, q_y, q_z$</td>
</tr>
<tr>
<td>Distributed forces</td>
<td>$p_x, p_y, p_z$</td>
</tr>
<tr>
<td>Natural frequency</td>
<td>$\omega$</td>
</tr>
</tbody>
</table>
1.82 Hz (≈ 4.1389 × 1.66078²/2π), which is higher than the primary forcing frequency 2/rev showing a stiff baseline design. The peak values of the concentrated loads transmitted from the rotor at the top of the tower were calculated at a rated wind speed of 8.0 m/s.

The associated maximum stress at the tower base and deflection at the top were found to be $0.268 \times 10^8$ N/m² and 0.02844 m, respectively, which are much less than the allowable values. On the other hand, considering critical loading conditions discussed above in Section 3.4, the maximum stress at tower base has reached a value of $0.6523 \times 10^8$ N/m² for the overload design condition with wind velocity $V_0 = 27$ m/s. The tip deflection increased to 0.0658 m.

5.3. Optimum tower solutions

In this section we shall consider, as a case study, tower optimization when subjected to a suddenly applied wind with a velocity equal to the survival value of 70.0 m/s. Two different optimum designs were obtained by considering two types of the objective function. The first type, type-1, considered maximization of the first three bending frequencies, $\Gamma_{bi}$, for the combined tower/rotor structure, whereas the second type, type-2, considered maximization of the frequencies of the tower alone. Fig. 9 presents the attained optimum tower designs for the ERDA-NASA MOD-0 wind turbine by implementing each of the above two formulations. As seen, the optimum tower configuration produced from the first type consists of six-modules while that from the second type consists of eight-modules, although the starting design was the same for both cases. It was found that the optimization algorithm treated the number of modules as an additional implicit variable. Sometimes the computer discarded one or more segments by letting their heights sink towards their lower limits (i.e. zeros), and sometimes made two consecutive modules identical, i.e. having the same diameter and thickness. It is also remarked that type-1 formulation results in a higher frequency
level than type-2, with the penalty of increasing structural mass. Both types produce substantial improvements in mass and stiffness as compared with those of the baseline design described in Table 5. Mass saving achieved from type-1 is shown to be about 7.6% and the calculated maximum deflection at tower top is 0.0393 m, which is much less than the corresponding value of 0.097 m of the baseline design. On the other hand, type-2 formulation produces a slightly softer design where mass saving has reached a value of 12.4% but with increased tip deflection to 0.0563 m.

The optimum values of the second mode frequencies obtained from type-1 were too close to 29/rev and 49/rev for the combined system and tower alone, respectively. Even though they are far away from the primary forcing frequencies (1/rev, 2/rev, 3/rev), they have been

---

**Fig. 9.** Optimum tower configurations of MOD-0 wind turbine (round tubular sections).
separated by applying the previously suggested mathematical concept. That is, instead of solving the eigenvalue problem in the natural frequencies, we choose the desired frequency and solve the eigenvalue problem in one of the design variables. The resulting modified optimum frequencies were given by:

\[
\begin{align*}
\text{Combined system} & : \quad \Gamma_{n} = 2.14344, 5.44208, 9.17620, \\
\text{Tower alone} & : \quad \Gamma_{n} = 4.53961, 29.26349, 83.19975.
\end{align*}
\]

which are seen to be well separated from the exciting frequencies.

Comparison between the present optimum solution and the original design of MOD-0 tower is given in Table 6. The frequency interference diagram (Campbell-diagram) is presented in Fig. 10, in which both the natural and exciting frequencies are normalized with respect to the rotor design speed, \( \Omega_0 = 4.2 \text{ rad/s} \).

### 6. Conclusions

Several optimization strategies for the structural design of wind turbine towers are developed and investigated through extensive computer implementations. The developed models have been successfully applied to an existing 100 kW wind turbine (ERDA-NASA, MOD-0), where optimum design trends were obtained through the use of the interior penalty function technique. It has been proved that maximization of a weighted-sum of the system natural frequencies is the most representative objective function which directly reflects the major design goals and ensure a balanced improvement in both mass and stiffness. Extensive computer experimentation has proved that the natural frequencies, even though implicit functions in the design variables, are well-behaved, monotonic and

---

**Table 6**

<table>
<thead>
<tr>
<th>Tower design</th>
<th>Fundamental frequency (Hz)</th>
<th>Mass saving</th>
<th>Maximum deflection ( m ), ( V_0 = 70 \text{ m/s} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original space truss</td>
<td>5.1</td>
<td>1.59, 1.76, 2.1 (different rotor positions)</td>
<td>–</td>
</tr>
<tr>
<td>Baseline cantilever</td>
<td>5.06</td>
<td>1.82</td>
<td>0.0</td>
</tr>
<tr>
<td>Optimum cantilever</td>
<td>7.92</td>
<td>3.06</td>
<td>7.60%</td>
</tr>
</tbody>
</table>

---

**Fig. 10.** Campbell frequency-diagram of combined tower/rotor. ERDA-NASA MOD-0 wind turbine (\( \Omega_0 = 40 \text{ rpm} \)).
defined every where in the design space. Global optimality can be attained from the proposed models and exact placement of the frequencies can be obtained by freely selecting their desired values. Appropriate nondimensionalization of the various parameters and variables throughout the problem formulation has led to a naturally scaled optimization model, which eliminates the need for scaling the design variables as usually suggested by similar optimization procedures. Another useful conclusion is the possibility of selection of the module height as a main design variable. This important variable is always missed by most of the previous research work dealing with structural optimization where the cross-sectional parameters are considered the only effective design variables. Finally, the present exact optimization analysis saves much of the computer time required by the finite-element and other discretized approximate methods.

References