NORMAL FORM TRANSFORMATION AND AN APPLICATION TO A FLUTTER-TYPE OF VIBRATION

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Abstract—Wake-induced oscillations affect bundled conductors of overhead transmission lines exposed to moderate to strong crosswinds. They represent a flutter-type of vibration that arises from the shielding effects of windward conductors on leeward ones. We consider a twin horizontal bundle of overhead lines. A mechanical model of this bundle is described by two cylinders mounted on springs, one in the wake of its neighbour. The critical wind velocity (bifurcation point) of incipient flutter is identified (Kern and Maitz, 1994, Proc. VII Int. Conf. on Boundary and Int. Layers Comp. and Asymptotic Methods, Beijing, China; Kern et al., 1995, SFB-Report 23, Institute of Mathematics, Graz). This bifurcation point yields a generalized Hopf bifurcation with non-semisimple double imaginary eigenvalues (case of 1:1 resonance). A non-linear analysis is adopted to investigate the post-bifurcational behaviour of the oscillations. In order to reduce the non-linear dynamical system to a simpler system without loss of generality of the dynamical behaviour, we use the normal form methods. In this method all non-linear terms which do not contribute to the dynamical behaviour are eliminated. Using a normal form transformation we get a three-dimensional system of amplitude equations which is analyzed qualitatively with regard to the unfolding parameters. © 1998 Elsevier Science Ltd. All rights reserved

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1. INTRODUCTION

The rising energy requirement and the associated need to increase the voltage of overhead lines (e.g. 380 kV) give rise to a number of problems which can be solved with the aid of bundled conductors. In the case of twin bundled conductors or quadruple bundles, two conductors lie in the horizontal plane, separated by a distance of 10 to 25 conductor diameters. Observations have been made indicating that under certain conditions, i.e. wind velocity and surrounding terrain, these bundles are aerodynamically unstable. These instabilities are due to the leeward conductor lying in the wake of the windward one. The instability mode has been shown to exhibit an elliptical path with the major axis inclined to the horizontal direction at a small angle. Using quasi-steady linear flutter theory, Price and Simpson [3–5] have predicted flutter boundaries for particular cases and have demonstrated these instabilities under laboratory conditions.

In the first part of this paper the equations of motion are generated. The stability of the equilibrium state of this system of differential equations is investigated in the following sections. Using Routh’s test function the critical wind velocity and the corresponding critical eigenvalues are determined. This linear model is a valuable tool in understanding some aspects of the conditions under which flutter is triggered, but it is known that the linear model is incapable of predicting the characteristics of the resulting limit cycle. This incompleteness of the linear model is eliminated by introducing a non-linear model which permits the investigation of the post-bifurcational behaviour. A normal form transformation is used in order to minimize the number of non-linear terms. Subsequently, the unfolding parameters are calculated by which the entire qualitative behaviour of the dynamical system can be investigated. In doing so we achieve a three-dimensional system of amplitude equations where the equilibria yield periodic motions, i.e. limit cycles, of the original dynamical system.

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2. THE EQUATIONS OF MOTION OF THE LEEWARD CONDUCTOR

For simplicity we only consider a horizontal twin bundle simulated by a rigid smooth cylinder mounted on springs in the wake of its identical, but fixed neighbour.

Then referring to Simpson [4] the equations of motion of the leeward conductor are given by

\begin{align*}
ml\ddot{x} + E_{xx}\dot{x} + E_{xz}\dot{z} &= F_x, \\
ml\ddot{z} + E_{zx}\dot{x} + E_{zz}\dot{z} &= F_z,
\end{align*}

where

\begin{align*}
F_x &= -\frac{1}{2}\rho c V_r \left( \tilde{C}_D(V + \dot{x}) - \tilde{C}_L\dot{z} \right), \\
F_z &= -\frac{1}{2}\rho c V_r \left( \tilde{C}_L(V + \dot{x}) + \tilde{C}_D\dot{z} \right),
\end{align*}

and \( E_{xx}, E_{zz} \) is the direct stiffness of the support system, \( E_{xz} = E_{zx} \) the static coupling, \( m \) the mass of conductor per unit length, \( l \) the conductor length, \( c \) the conductor diameter, \( \rho \) the density of air (1.225 kg/m\(^3\)), \( \dot{x}, \dot{z} \) the horizontal, vertical displacement of leeward conductor, \( V, V_r \) the local and relative local wind speed, \( b = V/V_r \) the rate of the local to free stream velocity, and \( \tilde{C}_D, \tilde{C}_L \) the drag and lift coefficient based on the local wind speed. With \( b^2 := 1.2/C_D(0, 0), b^2 = C_D/C_D = C_L/C_L \).

The coefficients \( \tilde{C}_D = \tilde{C}_D(\dot{x}, \dot{z}) \) and \( \tilde{C}_L = \tilde{C}_L(\dot{x}, \dot{z}) \) are non-linear polynomials adopted from Diana et al. [6] which are numerical approximations of wind tunnel tests. The functions of these polynomials are plotted in Figs (2) and (3). Here \( \tilde{x} = x_0 - x, \tilde{z} = z_0 - z \) is the relative distance of the leeward conductor to the windward one.

The introduction of the dimensionless variables

\begin{align*}
x &= \frac{x}{c}, \\
z &= \frac{z}{c}, \\
C_D &= b^2\tilde{C}_D, \\
C_L &= b^2\tilde{C}_L
\end{align*}

and using the fact that

\[ \tilde{V}_r = \sqrt{(V + \dot{x})^2 + \dot{z}^2} \]

lead to the system

\begin{align*}
\ddot{x} + \frac{E_{xx}}{ml} x + \frac{E_{xz}}{ml} z &= -\frac{\rho \sqrt{(bV + c\dot{x})^2 + c^2\dot{z}^2}}{2mb^2} \left\{ C_D(bV + c\dot{x}) - C_L\dot{z} \right\}, \\
\ddot{z} + \frac{E_{zx}}{ml} x + \frac{E_{zz}}{ml} z &= -\frac{\rho \sqrt{(bV + c\dot{x})^2 + c^2\dot{z}^2}}{2mb^2} \left\{ C_L(bV + c\dot{x}) + C_D\dot{z} \right\},
\end{align*}

where \( x, z \) are the horizontal and vertical dimensionless displacement of leeward conductor, \( V \) the free stream velocity, and \( C_D, C_L \) the drag and lift coefficient based on the free stream.
3. LINEARIZED ANALYSIS

We perform the usual stability analysis of non-linear differential equations, write equation (1) as a system of first-order differential equations by the transformation

\[ \begin{align*} x_1 &= x, \\
x_2 &= \ddot{x}, \\
x_3 &= z, \\
x_4 &= \dddot{z} \end{align*} \]

and determine the equilibrium \( x_e \) of the system

\[ \mathbf{X} = G(\mathbf{X}, V), \quad \mathbf{X} = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4, \]

where the wind velocity \( V \) is considered to be a parameter.

Using the Taylor series expansion about \( x_e \) and the transformation \( \mathbf{X} = \hat{\mathbf{X}} + x_e \) we get the new system

\[ \dot{\hat{\mathbf{X}}}(t) = A(V) \hat{\mathbf{X}}(t) + \hat{\mathbf{F}}(\hat{\mathbf{X}}(t), V), \]

where

\[ A(V) = \left. \frac{\partial G_i}{\partial X_j} \right|_{x=x_e} \]

and \( \hat{\mathbf{F}}(\hat{\mathbf{X}}, V)(t) \) are the higher-order terms.

The stability of the trivial solution \( \hat{\mathbf{X}} = 0 \) is determined by the eigenvalues of the linearized system. Therefore, the characteristic equation of \( A(V) \) has the shape

\[ \lambda^4 + p_3\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 = 0. \]
Relating to ref. [7] flutter will occur if the wind speed exceeds the critical value at which Routh’s test function $T_3$ vanishes.

If we drop all aerodynamic damping terms $(\dot{x}, \dot{z})$ in equation (1) $A(V)$ simplifies to

$$A(V) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & a_1(V) & 0 & a_2(V) \\
0 & 0 & 0 & 1 \\
b_1(V) & 0 & b_2(V) & 0
\end{pmatrix}.$$  \hfill (4)

In this case $p_3 = p_1 = 0$ and $T_3 \equiv 0$, thus we have to solve the $T_2$-condition:

$$T_2 := p_2^2 - 4p_0 = 0.$$  

By solving this equation with respect to $V$ we get the critical wind speeds. Between these critical values the imaginary parts of $\lambda$ coalesce which is a necessary condition for the flutter phenomenon.

Let $V_c$ denote the critical wind velocity for incipient flutter computed by the linearized model. Now there exists a real transformation matrix $T$ with the property

$$T^{-1}A(V_c)T = P,$$  \hfill (5)

where

$$P = \begin{pmatrix}
0 & -\omega & 1 & 0 \\
\omega & 0 & 0 & 1 \\
0 & 0 & 0 & -\omega \\
0 & 0 & \omega & 0
\end{pmatrix}.$$ \hfill (6)

Thus, the transformation

$$\tilde{X} = TX$$

yields the new system

$$\dot{\tilde{X}} = PX + F(X).$$ \hfill (7)

The trivial solution of the linear system $\dot{\tilde{X}} = PX$ is unstable and therefore, we also have to consider the non-linear coefficients. The complexity of the non-linear terms makes the calculation too difficult. Therefore, we reduce the system to only those terms which contribute to the dynamical behaviour. This can be done by the aid of the theory of normal form transformations [8–10].

4. NORMAL FORM OF THE 1:1 RESONANCE HOPF BIFURCATION

Now the method of normal form transformation is applied to our system (7) at the flutter instability region. According to equations (6) and (7) we consider the following dynamical system:

$$\dot{\tilde{X}} = \begin{pmatrix}
0 & -\omega & 1 & 0 \\
\omega & 0 & 0 & 1 \\
0 & 0 & 0 & -\omega \\
0 & 0 & \omega & 0
\end{pmatrix} \tilde{X} + F(X)$$ \hfill (8)

with

$$X = \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix}, \quad F \in \mathbb{R}^4.$$
Using the complex variables

\[ y_1 = x_1 + ix_2, \quad y_2 = x_3 + ix_4, \]

equation (8) changes to

\[
\begin{pmatrix}
\dot{y}_1 \\
\dot{y}_2 \\
\dot{y}_1 \\
\dot{y}_2
\end{pmatrix} = \begin{pmatrix}
i\omega & 1 & 0 & 0 \\
0 & i\omega & 0 & 0 \\
0 & 0 & -i\omega & 1 \\
0 & 0 & 0 & -i\omega
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2 \\
y_1 \\
y_2
\end{pmatrix} + \begin{pmatrix} f_1 \\
f_2 \\
f_1 \\
f_2
\end{pmatrix},
\tag{9}
\]

where \( f_1 = F_1 + iF_2, \ f_2 = F_3 + iF_4. \) Since the last two equations in equation (9) are the conjugate complex counterpart of the first two we only need to consider the truncated system

\[
\dot{Y} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ 0 \\ 0 \end{pmatrix} = J\dot{Y} + G(Y).
\tag{10}
\]

In order to get the normal form of the dynamical system we use the transformation

\[
\dot{Y} = z + h(z, \bar{z})
\tag{11}
\]

with

\[
h(z, \bar{z}) = \begin{pmatrix}
\sum_{ijkl} h_{ijkl} z_1^i z_2^j z_1^k z_2^l \\
\sum_{ijkl} h_{ijkl} z_1^i z_2^j z_1^k z_2^l
\end{pmatrix}
\tag{12}
\]

and determine the non-linear terms which contribute to the solution of the system (7) by the aid of the following two steps:

**Step 1:** First, we only consider the simplified matrix

\[
\begin{pmatrix}
i\omega & 0 \\
0 & i\omega
\end{pmatrix}
\tag{13}
\]

and from equation (13) we determine the resonance cases. The resonance condition (see ref. [9]) yields

\[
i\omega = i\omega(i - j + k - l),
\]

or

\[
i - j + k - l = 1 \quad \text{and} \quad i + j + k + l \geq 2.
\tag{14}
\]

From equation (14) it follows that the only interesting non-linear terms are the ones given in Table 1 and consequently we use in equation (12) the following simplified \( h' \):

\[
h_i = h_{1100} z_1^2 \bar{z}_1^2 + h_{2001} z_2^2 \bar{z}_2 + h_{1110} z_1 \bar{z}_1 z_2,
\]

\[
\quad + h_{1011} z_1 \bar{z}_2 z_2 + h_{1010} z_1^2 \bar{z}_2^2 + h_{0021} z_1 \bar{z}_1 z_2^2, \quad i = 1, 2.
\tag{15}
\]

The associated space of vector-valued monomials of degree 3 is

\[
H_3 = \text{span}\left\{ \begin{pmatrix} |z_1|^2 z_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ |z_1|^2 z_1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} |z_1|^2 z_2 \\ 0 \\ |z_2|^2 z_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ |z_1|^2 z_2 \\ 0 \\ |z_2|^2 z_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ |z_1|^2 z_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ |z_1|^2 z_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ |z_1|^2 z_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ |z_1|^2 z_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ |z_1|^2 z_2 \end{pmatrix} \right\}.
\]

**Step 2:** Inserting the transformation (11) with (15) into equation (10) i.e.,

\[
\dot{y}_1 = i\omega y_1 + y_2 + f_1,
\]

\[
\dot{y}_2 = i\omega y_2 + f_2,
\]
results in the algebraic system

\[- \text{ad}L(h_1) - h_2 = f_1,\]

\[- \text{ad}L(h_2) = f_2,\]

where the matrix \(- \text{ad}L\) is a matrix representation of the Lie-operator (see ref. [10]) and is equal to

\[
- \text{ad}L = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}.
\]

Hence, we have to solve

\[
\begin{pmatrix}
- \text{ad}L & - I \\
0 & - \text{ad}L
\end{pmatrix}
\begin{pmatrix}
h_1 \\
h_2
\end{pmatrix}
= \begin{pmatrix}
f_1 \\
f_2
\end{pmatrix}.
\]

This system does not have a unique solution if \(f \in \text{Ker}(C^T)\), because from linear algebra it is well known that each linear vector space \(H^k\) can be composed of the range of \(C\) and the kernel of the adjoint of \(C\)

\[H^k = R(C) \oplus \text{Ker}(C^T).\]

Therefore, all terms of \(f_i\) which solve the system

\[C^Tf = \begin{pmatrix}
- (\text{ad}L)^T & 0 \\
- I & - (\text{ad}L)^T
\end{pmatrix}
\begin{pmatrix}
f_1 \\
f_2
\end{pmatrix}
= 0
\]

cannot be eliminated in equation (9).

Here, the range of \(C\) is set up by

\[
R(C) = C(H_3) = \text{span}\left\{\binom{z_1^2z_2 + 2|z_1|^2z_2}{0}, \binom{|z_1|^2z_1}{0}, \binom{|z_2|^2z_2}{0}, \binom{z_1^2z_2}{0}, \binom{0}{z_2^2z_2}, \binom{|z_1|^2z_1}{0}, \binom{z_2^2z_2}{0}, \binom{-z_1^2z_2}{0}, \binom{-2z_1|z_2|^2}{0}, \binom{z_1^2z_2}{0}, \binom{-z_1|z_2|^2}{0}, \binom{z_2^2z_2}{0}\right\}
\]
and the kernel of $C^T$ consists of

$$
\text{Ker}(C^T) = \text{span}\left\{\begin{array}{cccc}
0 & 0 & 2 & 0 \\
2 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right\}.
$$

Hence, the vectors are

$$
\begin{align*}
\dot{z}_1 &= \imath \omega z_1 + z_2 + \mathcal{O}(|z|^5) \\
\dot{z}_2 &= \imath \omega z_2 + \tilde{a}|z_1|^2 z_1 + \tilde{b}(2\tilde{z}_1^2 \tilde{z}_2 - |z_1|^2 z_2) + \tilde{c}\tilde{z}_1 z_2^2 + \tilde{d}(|z_1|^2 z_2 + z_1^2 \tilde{z}_2) + \mathcal{O}(|z|^5)
\end{align*}
$$

and this yields the following normal form of system (8):

$$
\begin{align*}
\dot{\tilde{z}}_1 &= \imath \omega \tilde{z}_1 + \tilde{z}_2 + \mathcal{O}(|\tilde{z}|^5) \\
\dot{\tilde{z}}_2 &= \imath \omega \tilde{z}_2 + a|\tilde{z}_1|^2 \tilde{z}_1 + b\tilde{z}_1^2 \tilde{z}_2 + c\tilde{z}_1^2 \tilde{z}_2 + d|\tilde{z}_1|^2 \tilde{z}_2 + \mathcal{O}(|\tilde{z}|^5)
\end{align*}
$$

respectively, where $a, b, c, d \in \mathbb{C}$. Since many steps are needed to obtain this normal form, the coefficients $a, b, c, d$ consist of a lengthy combination of the Taylor series coefficients of $\tilde{F}$ in equation (3) up to third order. For details see ref. [11]. Therefore, it is difficult to obtain a physical interpretation of the coefficients.

### 5. UNFOLDINGS OF THE SYSTEM

We determine the unfoldings by starting with equation (3) and making use of the suspension trick following Troger et al. [9]. Therefore, we add the equation $\dot{\varepsilon} = 0$, where $\varepsilon = V - V_c$. This results in

$$
\dot{\tilde{X}} = \Lambda(\varepsilon)\tilde{X} + \tilde{F}(\tilde{X}, \varepsilon),
$$

$$
\dot{\varepsilon} = 0.
$$
Expanding $\mathbf{A}(\varepsilon)$ into a series in $\varepsilon$ and transforming equation (18) by the transformation matrix $T$ used in equation (5) and additionally, by the transition to complex variables we obtain

\[
\begin{align*}
\hat{y}_1 &= i\omega y_1 + y_2 + \varepsilon(x_1 y_1 + \beta_1 y_2 + \gamma_1 \hat{y}_1 + \delta_1 \hat{y}_2) + \text{h.o.t.}, \\
\hat{y}_2 &= i\omega y_2 + \varepsilon(x_2 y_1 + \beta_2 y_2 + \gamma_2 \hat{y}_1 + \delta_2 \hat{y}_2) + \text{h.o.t.},
\end{align*}
\]

(19a)

\[
\begin{align*}
\hat{y}_1 &= -i\omega y_1 + y_2 + \varepsilon(x_1 y_1 + \beta_1 y_2 + \gamma_1 \hat{y}_1 + \delta_1 \hat{y}_2) + \text{h.o.t.}, \\
\hat{y}_2 &= -i\omega y_2 + \varepsilon(x_2 y_1 + \beta_2 y_2 + \gamma_2 \hat{y}_1 + \delta_2 \hat{y}_2) + \text{h.o.t.},
\end{align*}
\]

(19c)

\[
\hat{\dot{y}}_2 = 0,
\]

(19e)

where $x_j, \beta_j, \gamma_j, \delta_j \in \mathbb{C}$ for $j = 1, 2$. Now we can apply the normal form transformation to equations (19a)-(19e). The resonance condition for equations (19a) and (19b) is

\[i\omega(i - j + k - l) + m(0) - i\omega = 0,
\]

which has been changed because of fifth eigenvalue $0$ of the equation (19e). This gives the solution $i = 1, j = 0, k = 0, l = 0, m = 1$ and $i = 0, j = 0, k = 1, l = 0, m = 1$. Analogous to Steps 1 and 2 in Section 4 we get the normal form

\[
\begin{align*}
\hat{z}_1 &= i\omega z_1 + z_2 + \varepsilon A z_1 + \mathcal{O}(|z|^5), \\
\hat{z}_2 &= i\omega z_2 + \varepsilon A z_2 + \varepsilon B z_1 + a|z_1|^2 z_1 + b z_1^2 \bar{z}_2 + c \bar{z}_1 z_2^2 + d|z_1|^2 z_2 + \mathcal{O}(|z|^5).
\end{align*}
\]

(20)

Here additional linear terms in $z_1$ and $z_2$ exist, which can be viewed as the versal unfolding parameter. Using a rescaling of time and $z_2$ to keep the imaginary part equal to $i$ we can rewrite the unfolding system (20) to

\[
\begin{align*}
\hat{\dot{z}}_1 &= (i + \lambda) z_1 + z_2 + \text{h.o.t.} \\
\hat{\dot{z}}_2 &= \mu z_1 + (i + \lambda) z_2 + \text{h.o.t.}
\end{align*}
\]

(21)

where $\lambda = (x, \mu_1, \mu_2), x \in \mathbb{R}$ and $\mu = \mu_1 + i\mu_2 \in \mathbb{C}$ are now the universal unfolding parameters of codimension 3 system, which also corresponds to the results of ref. [12].

6. APPLICATION TO OUR SYSTEM

Finally, the evaluations of Sections 4 and 5 are performed on the special form of $\mathbf{A}(V)$ as given in equation (4). Using the transformation matrix

\[
T = \begin{pmatrix}
0 & b_2 + \omega^2 & -2\omega & 0 \\
\omega(b_2 + \omega^2) & 0 & 0 & b_2 + 3\omega^2 \\
0 & -b_1 & 0 & 0 \\
-\omega b_1 & 0 & 0 & -b_1
\end{pmatrix}
\]

and calculating the normal form and unfoldings in accordance with Sections 4 and 5 we get the system

\[
\begin{align*}
\hat{\dot{z}}_1 &= (i + \alpha) z_1 + z_2 + \mathcal{O}(|z|^5) \\
\hat{\dot{z}}_2 &= \mu z_1 + (i + \alpha) z_2 + a|z_1|^2 z_1 + b z_1^2 \bar{z}_2 + c \bar{z}_1 z_2^2 + d|z_1|^2 z_2 + \mathcal{O}(|z|^5),
\end{align*}
\]

(22)

where $\alpha = 0, \mu \in \mathbb{R}, a, c \in \mathbb{R}$ and $b, d$ are purely imaginary. The ranges for the parameters originate from the fact that the aerodynamic damping terms are neglected. The main influence of these restrictions to the coefficients is that there is no unique solution for $w_2$ in system (25).

Therefore, in this special case only one mathematical parameter $\mu$ remains which represents the physical parameter $V$ through $\varepsilon$ in the following way

\[
\mu = \frac{\varepsilon \text{Re}(B)}{(\omega + \varepsilon \text{Im}(A))^2}
\]
with
\[ A = \frac{1}{2} a_1 + \frac{1}{2} \gamma_2, \quad B = a_2, \]
where \( a_1, a_2, \) and \( \gamma_2 \) are derived from equation (19). From the system matrix \( A(V) \) we obtain \( \mu > 0 \) as \( v > 0 \) or \( V > V_c \).

Note that \( z_1 = 0 \) implies \( z_2 = 0 \) in equation (22). This suggests that we seek solutions for which \( z_1 \) is a factor of \( z_2 \), that is,
\[ z_2 = z_1 w = z_1 (w_1 + iw_2). \]

Changing to polar coordinates for \( z_1 \) by
\[ z_1 = r e^{i \phi}, \quad z_2 = r e^{i \phi} w \]
and considering that the normal form is odd in \( r \), i.e. has \( \mathbb{Z}_2 \) symmetry, we make the transformation \( s = r^2 \). This yields the system
\[ \dot{s} = 2sw_1 + \text{h.o.t.}, \]
\[ \dot{w}_1 = \mu_1 + as - w_1^2 + w_2^2 + (\text{Im}(b) - \text{Im}(d))sw_2 + cs(w_1^2 - w_2^2) + \text{h.o.t.}, \tag{23} \]
\[ \dot{w}_2 = -2w_1 w_2 + (\text{Im}(b) + \text{Im}(d))sw_1 + 2csw_1 w_2 + \text{h.o.t.}, \]
\[ \dot{\phi} = 1 + w_2 + \text{h.o.t.}. \tag{24} \]

Since locally \( \phi \approx 1 \) in equation (24), the relative equilibria of equation (23) are periodic orbits of equation (22). Therefore, we confine our further investigation to the calculation of the equilibria of the system (23), which are the roots of the following system:
\[ 0 = 2sw_1, \tag{25a} \]
\[ 0 = \mu_1 + as - w_1^2 + w_2^2 + (\text{Im}(b) - \text{Im}(d))sw_2 + cs(w_1^2 - w_2^2), \tag{25b} \]
\[ 0 = -2w_1 w_2 + (\text{Im}(b) + \text{Im}(d))sw_1 + 2csw_1 w_2. \tag{25c} \]

For \( s > 0 \) we get from equation (25a) that \( w_1 = 0 \) and from equation (25b) we can deduce
\[ s = -\frac{\mu + w_2^2}{a + (\text{Im}(b) - \text{Im}(d))w_2 - cw_2^2} > 0. \]

Since \( w_2 \) starts at 0 and is considered in the neighbourhood of 0, the first equilibrium is at \( w_2 = 0 \) with
\[ s = -\frac{\mu}{a}. \]

In order to obtain a limit cycle we have to demand that \( s > 0 \) as the radius \( r = \sqrt{s} \). Since the mathematical unfolding parameter \( \mu \) is strictly positive for the wind velocity \( V > V_c \) there will only be a stable limit cycle if \( a < 0 \). The other coefficients do not influence this limit cycle.

Thus, we only have an oscillation in the \( z_1 \)-plane at \( z_2 = 0 \). At equilibria with \( w_2 \neq 0 \), the resulting limit cycle is slightly different. However, only the amplitude and phase and not the bearing are affected. With \( w_2 = 0 \) our numerical experiments have shown a very good fit of bearing and amplitude with respect to the numerical simulation.

6.1. Numerical example

In our numerical example the horizontal distance between the windward and leeward conductor is \( x_0 = 12 \) (in conductor diameter) and the vertical distance is \( z_0 = -1.4 \) (also in conductor diameter and the negative value means that the leeward conductor lies below the windward one).

The other system parameters are:
- Conductor diameter \( c = 0.03048 \) m, “active length” of the moving cylinder \( l = 45.72 \) m, mass \( m = 1.596 \) kg/m and the first and second natural frequency of the conductor \( \omega_1 = 8.34 \) and \( \omega_2 = 0.91 \omega_1 \) rad/s. Then the critical wind velocity \( V_c \) lies at 8.675837 m/s and the corresponding equilibrium of the original system (2) is \( x_c = -0.3015 \) and \( z_c = -0.1219 \) when the leeward conductor without wind is at the origin. With these parameters we get the
following solutions. In Fig. 4 we have plotted the numerical solution of the original equation computed by the computer algebra system MAPLE with the standard Runge–Kutta – 4/5 procedure. This calculation needs about 1300 CPU seconds on a DEC Alpha Workstation. Figure 5 shows the maximum solution computed by a MAPLE-program using the normal form transformation described above. This calculation only needs about 6 CPU seconds on the same workstation. The solutions are computed at a wind velocity, which only lies 0.01 m/s above the critical one.

7. CONCLUSIONS

Using this non-linear investigation together with the quasi-steady state theory proposed in this paper we can analytically determine the flutter zones and the corresponding critical wind velocity for incipient flutter. At this critical velocity we have obtained from the non-linear model amplitude equations which can be solved analytically by the aid of the normal form transformation. Analysing the amplitude equations, the radius of the resulting limit cycle can be determined. This radius yields the maximum amplitude of the limit motion by the back transformation. This maximum amplitude is almost identical to that achieved by numerical methods on the one hand, as seen in the comparison of Fig. 4 with Fig. 5, but the amount of CPU-time is significantly less than that of the numerical solution on the other hand. Therefore, this method supplies good results for our problem.

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